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Journal of Functional Analysis 230 (2006) 116–168

JOURNAL OF Functional Analysis

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High-frequency limit of the Helmholtz equation with variable refraction index

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Received 2 December 2004; accepted 1 June 2005

Communicated by M. Christ

Available online 3 August 2005

Abstract

We study the high-frequency limit of the Helmholtz equation with variable refraction index and a source term concentrated near a p-dimensional affine subspace. Under some conditions, we first derive uniform estimates in Besov spaces for the solutions. Then, we prove that the semi-classical measure associated with these solutions satisfies the stationary Liouville equation with an explicit source term and has certain radiation property at infinity.

Keywords: Helmholtz equation; Semi-classical measure; High frequency limit; Uniform resolvent estimates; Schroedinger operator; Liouville equation

1. Introduction

In this work, we are interested in the high-frequency limit of the Helmholtz equation with a source supported near a smooth submanifold $\Gamma \subset \mathbb{R}^d$:

$$i\,\frac{\alpha_\varepsilon}{\varepsilon}\,u_\varepsilon + \Delta u_\varepsilon + \frac{n^2(x)}{\varepsilon^2}\,u_\varepsilon = \mathcal{S}_\varepsilon(x), \quad x \in \mathbb{R}^d, \tag{1.1}$$

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doi:10.1016/j.jfa.2005.06.003

where $n(x)$ is the refraction index, ε is a small parameter, and α_ε is a positive regularizing parameter with $\alpha_\varepsilon \to \alpha \geqslant 0$ as $\varepsilon \to 0$, and the source term $\mathcal{S}_\varepsilon$ is of the form

$$\mathcal{S}_\varepsilon(x) = -\varepsilon^{-q} \int_\Gamma e^{i\phi(y)/\varepsilon} A(y) S\left(\frac{x-y}{\varepsilon}\right) d\sigma(y). \tag{1.2}$$

Here Γ is a p-dimensional submanifold of $\mathbb{R}^d$ and q a normalizing index. In this work, we assume that $\Gamma = \mathbb{R}^p \times \{0\} \subset \mathbb{R}^d$ with $1 \leqslant p < d$, A and S are smooth functions, with A being of compact support and S rapidly decaying at infinity. When Γ is not flat, there may be some additional difficulties related to the geometry of Γ in the uniform estimate of u_ε, since then we cannot reformulate (1.1) to the form of (3.2). We shall use in some steps microlocalization in terms of the phase ϕ and calculus of pseudo-differential operators. Thus, we assume that $\phi(y)$ and $n(x)$ are real smooth functions satisfying that

$$|\partial_y^\beta \phi(y)| \leqslant C_\beta \langle y\rangle^{1-|\beta|}, \quad \forall \beta \in \mathbb{N}^p,\ y \in \mathbb{R}^p, \tag{1.3}$$

and there exists some $\rho_0 > 0$ such that

$$|\partial_x^\beta n^2(x)| \leqslant C_\beta \langle x\rangle^{-|\beta|-\rho_0}, \quad \forall \beta \in \mathbb{N}^d, \beta \neq 0,\ x \in \mathbb{R}^d. \tag{1.4}$$

The conditions needed in each result are less restrictive and will be stated separately.

The solution u_ε to (1.1) determines the field of a light source in an inhomogeneous medium with refraction index $n(x)$, wave length ε and the source emits waves with amplitude A and phase ϕ, oscillating with the same wave length ε as u_ε, so as to create resonant effects between the highly oscillating functions u_ε and the source itself.

According to the relation between the size of $|\nabla^\tau \phi|$ and n, the authors of [5] introduced three regimes: propagative regime if $|\nabla^\tau \phi(y)| < n(y)$, $\forall y \in \Gamma$; characteristic regime if $|\nabla^\tau \phi(y)| = n(y)$, $\forall y \in \Gamma$; and resonant regime if $|\nabla^\tau \phi(y)| > n(y)$, $\forall y \in \Gamma$. They determined the normalizing index q and formally derived the associated Liouville equation in each regime. In particular, in the propagative regime where $q = \frac{3+d+p}{2}$, they rigourously proved in [5] for constant refraction index the existence of a semi-classical measure (equally called Wigner measure), f, associated with the solutions of (1.1) and determined its Liouville equation which is of the form

$$+\alpha f(x,\xi) + \xi \cdot \nabla_x f + \tfrac{1}{2} \nabla_x n^2(x) \cdot \nabla_\xi f = Q(x,\xi). \tag{1.5}$$

Eq. (1.5) describes the propagation of energy in the medium with refraction index $n(x)$ as in geometric optics, and in this picture $f(x,\xi)$ can be seen as the energy density carried by rays located at the position x with velocity ξ. One may see [2,5,12] for more physical explanations. Actually by Theorem 5.1 of this work, under the assumption of some non-trapping property of the classical trajectories, the solution to (1.5) is given by

$$f(x,\xi) = \int_0^\infty e^{-\alpha s} Q(X^{-s}(x,\xi), \Xi^{-s}(x,\xi))\, ds, \tag{1.6}$$

in a weak sense, where $(X^s(x,\xi), \Xi^s(x,\xi))$ is the solution to the Hamiltonian system associated to $(|\xi|^2 - n^2(x))/2$

$$\begin{cases} \dfrac{d}{ds} X^s(x,\xi) = \Xi^s(x,\xi), & X^0(x,\xi) = x, \\ \dfrac{d}{ds} \Xi^s(x,\xi) = \dfrac{1}{2} \nabla_x n^2(X^s(x,\xi)), & \Xi^0(x,\xi) = \xi \end{cases} \tag{1.7}$$

with initial data (x,ξ) satisfying $|\xi|^2 - n^2(x) = 0$.

When Γ is one point, this problem is studied in [2] by making use of Morrey–Campanato-type estimate [15]. The source term in Liouville equation (1.5) was only conjectured there and this conjecture is proved by Castella [4]. When Γ is a p-dimensional affine subspace with $p \geqslant 1$, Morrey–Campanato-type estimate of Perthame and Vega [15] cannot be used as the source term does not decay fast enough in Γ direction. In this case, the authors of [5] treated the case of a constant refraction index $n(x) = n_0$. As commented by these authors in [5], the restriction on the refraction index is not just a technicality, it is linked to the difficulties in establishing uniform a priori estimates and in finding a reasonable "radiation condition at infinity".

While the steps of the present work are the same as in [5], there are two major differences. The first one is that when the refraction index is variable, the explicit calculation by Fourier transform used in [5] to obtain a priori estimates does not work. Although Eq. (1.1) looks like a semi-classical Schrödinger equation associated with $\varepsilon^2\Delta_x - n(x)^2$, in order to take into account the concentration effect of the source term, one has to work with the operator $\Delta_x - n(\varepsilon x)^2$. In terms of these new variables, the source term $\mathcal{S}_\varepsilon$ in (1.1) behaves asymptotically like

$$S_\varepsilon(\varepsilon x) \sim -\varepsilon^{-\frac{d-p+3}{2}} C e^{i\phi(\varepsilon x_1)/\varepsilon} S(0, x_2), (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{d-p}, \quad \varepsilon \to 0,$$

which has no decay in x_1 variables. As the limiting absorption principle always requires appropriate weight in full variables, the first difficulty in the present work is to overcome the lack of decay in Γ-direction. In order to do so, we are going to use the Mourre method and microlocal-resolvent estimates depending on a parameter to prove uniform a priori bound on u_ε and $w^\varepsilon_{z_1}$ defined below. Note that the role of microlocal-resolvent estimates is to distinguish different directions of oscillation and to compensate the lack of decay in some directions. The second difference is related to the determination of the source term in the limiting Liouville equation. We study as in [5]

$$w^\varepsilon_{z_1}(y) = \varepsilon^{\frac{(d-p-1)}{2}} u_\varepsilon(z_1 + \varepsilon y_1, \varepsilon y_2) e^{-\frac{i\phi(z_1)}{\varepsilon}}, \quad z_1 \in \Gamma, \tag{1.8}$$

which satisfies the equation

$$(-\triangle_y - n(z_1 + \varepsilon y_1, \varepsilon y_2)^2 - i\varepsilon\alpha_\varepsilon) w^\varepsilon_{z_1}(y) = W^\varepsilon_{z_1} \tag{1.9}$$

with

$$W^\varepsilon_{z_1}(y) = \int_{R^p} e^{i\frac{\phi(z_1+\varepsilon Z_1)-\phi(z_1)}{\varepsilon}} A(z_1 + \varepsilon Z_1) S(y_1 - Z_1, y_2)\, dZ_1. \tag{1.10}$$

To determine explicitly the source term in the Liouville equation, one need to prove that $w^\varepsilon_{z_1}$ converges weakly to the outgoing solution of the limiting equation

$$(-\triangle_y - n(z_1, 0)^2 - i0) w_{z_1}(y) = W_{z_1}, \tag{1.11}$$

where W_{z_1} is the pointwise limit of $W^\varepsilon_{z_1}$. This is highly non-trivial for variable refraction index. In fact, even in the case of point source with variable refraction index (i.e., Γ is reduced to one point $\{0\}$), this result was conjectured in [2] and is recently proved by Castella in [4] under an assumption on the dimension of self-intersections of classical Hamiltonian flow. When specified to the case of point source, our approach here gives an alternative proof of the conjecture of Benamou et al. [2] but without the additional assumption in [4].

In this work, we use two sets of assumptions: (A.1)–(A.3) to obtain the uniform estimate for u_ε in $B^*_{\frac{1}{2}}(\mathbb{R}^d)$, and (1.3), (1.4) and (A.4)–(A.6) to prove the weak convergence of $w^\varepsilon_{z_1}$ to w_{z_1}. (A.2) can be regarded as a combination of (A.4) and (A.5) with $z_1 = 0$ while (A.6) for some $\delta_0 > 0$ uniformly in all z_1 and y implies (A.4). We present them separately because some intermediate results hold under less restrictive conditions. In particular, the existence of the semi-classical measure is proved for generalized N-body-type refraction index. We need the conditions (A.5) and (A.6) for each z_1 in the passage of limit from (1.9) to (1.11). In the final step, we mainly use the condition (A.5) with $z_1 = 0$ to prove Theorem 5.1. Remark that (2.16), (A.5) (and also implicitly, (A.2)) are some kind of virial conditions, which are stronger than the general non-trapping condition (see (2.3)). These conditions allow us to use $A = (x \cdot D + D \cdot x)/2$ as conjugate operator in the Mourre method. Note that this operator is invariant under the change of scales and natural in the study of concentration phenomenon. Under the general non-trapping condition, one can still construct a family of conjugate operators [6]. But the lack of uniform control in ε in this general case prevents us from proving Theorem 3.3 which plays an important role in this work. In the case where Γ is just one point $x = 0$, if the three estimates of Theorem 3.3 are true for $z_1 = 0$, one can easily pass to the limit from (1.9) to (1.11) for $z_1 = 0$ without any other assumption on the classical flow associated with $\xi^2 - n(x)^2$. Finally, let us indicate that even when Γ is just one point, the existence of a semi-classical measure is only proved under the virial condition. See [2] and Remark 2.7. An interesting open question is to give a necessary and sufficient condition on the classical flow of $\xi^2 - n(x)^2$ such that the solution

$\{u_\varepsilon\}$ of (1.1) is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, which implies the existence of a semi-classical measure associated with $\{u_\varepsilon\}$ (see [3,8]).

The outline of the paper is the following: we first present some semi-classical resolvent estimates in Besov spaces in §2 and make a remark on parameter-dependent Morrey–Campanato-type estimate. The result of §2 is not directly used in this paper, however, in §3, we repeatedly use the same ideas and some microlocal resolvent estimates to achieve some a priori estimates for the Helmholtz equation. We study in §4 the weak limit of $w^\varepsilon_{z_1}$. Finally, we determine in §5 the Liouville equation satisfied by the semi-classical measure, calculate its source term and prove a weak outgoing radiation property at infinity.

As conventions in this work, we will denote the $L^2(\mathbb{R}^d)$ norm of a function and the operator norm by $\|\cdot\|$, C a positive constant which may be different in each line, and $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$.

2. Semi-classical resolvent estimates in Besov spaces

We want to show that the Mourre method can be used to obtain uniform estimates in Besov spaces for resolvent of operators depending on a small parameter. This idea goes back to Mourre [13,14] and was used in [10,22] for operators without small parameter. It is well-known that it can be used to prove semi-classical resolvent estimates in weighted L^2 spaces [6]. In this section, we shall prove semi-classical resolvent estimates in Besov spaces following the method of Mourre [14] and checking the dependence on the small parameter.

Eq. (1.1) can be rewritten as a semi-classical Schrödinger equation

$$(P(h) - (E + i\kappa))u_h = S^h(x), \tag{2.1}$$

where $h = \varepsilon$, $u_h = u_\varepsilon$, $S^h = -\varepsilon^2 \mathcal{S}_\varepsilon$, $\kappa = \varepsilon\alpha_\varepsilon$, and

$$P(h) = -h^2\Delta + V(x), \quad V(x) = E - n^2(x), \quad E > 0.$$

We want to give an estimate uniform in $h, \kappa \in]0, 1]$ for the resolvent

$$R(z, h) = (P(h) - z)^{-1}, \quad z = E + i\kappa,$$

in Besov spaces. When the refraction index $n(x)$ satisfies $n(x) = n_0 + O(\langle x\rangle^{-\varepsilon})$, as $|x| \to \infty$, one has $V(x) = O(\langle x\rangle^{-\varepsilon})$. $P(h)$ is a two-body Schrödinger operator. If $n(x) = n_1(x_1) + n_2(x_2)$ with $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d_1 + d_2 = d$, such that $n_j(x_j) = n_{0,j} + O(\langle x_j\rangle^{-\varepsilon})$ with $E = n^2_{0,1} + n^2_{0,2} > 0$, then $P(h)$ is a three-body Schrödinger operator. In the subsequence, we shall prove results for generalized N-body Schrödinger operators, hoping that they may be useful to study the Helmholtz equation with refraction index of the form

$$n(x) = \sum_{j=1}^{M} n_j(x_j), \quad n_j(x_j) = n_{j,0} + o(1), \quad |x_j| \to \infty, \tag{2.2}$$

where $n_{j,0} \geqslant 0$ with $E = \sum_j n_{j,0}^2 > 0$, and $x_j \in \mathbb{R}^{d_j} \subseteq \mathbb{R}^d$. Indeed, Theorem 4.1 for the limiting semi-classical measure is valid in such case.

To prove semi-classical resolvent estimates for two-body Schrödinger operators, it is well-known (see [16,18]) that the non-trapping condition on the classical Hamiltonian system is both necessary and sufficient. Recall that the energy $E > 0$ is called non-trapping for the classical Hamiltonian $p(x, \xi) = |\xi|^2 + V(x)$ if

$$\lim_{|t| \to \infty} |x(t; y, \eta)| = \infty, \quad \forall (y, \eta) \in p^{-1}(E). \tag{2.3}$$

Here, $(x(t; y, \eta), \xi(t; y, \eta))$ is the solution of the classical Hamilton system associated with $p(x, \xi)$

$$\begin{cases} \frac{\partial x}{\partial t} = \partial_\xi p(x, \xi), & x(0; y, \eta) = y, \\ \frac{\partial \xi}{\partial t} = -\partial_x p(x, \xi), & \xi(0; y, \eta) = \eta. \end{cases}$$

Under the condition (2.3), it is well-known that for any $s > \frac{1}{2}$

$$\|\langle x\rangle^{-s} R(z, h) \langle x\rangle^{-s}\| \leqslant C_s h^{-1}, \quad \forall h, \kappa \in]0, 1].$$

We shall prove semi-classical resolvent estimates in Besov space for generalized N-body Schrödinger operators.

Let us recall some notation for generalized N-body Schrödinger operators. Let $\mathbb{X}$ be a d-dimensional Euclidean space equipped with a quadratic form $q(\cdot)$. To simplify notation, we assume that $q(\cdot)$ is the canonical form on $\mathbb{X} = \mathbb{R}^d$. Let $\mathcal{A}$ denote the set of all cluster decompositions of an N-body system labeled by $\{1, 2, \ldots, N\}$, which are all possible partitions of the set $\{1, 2, \ldots, N\}$. To each $a \in \mathcal{A}$, it is assigned a subspace $\mathbb{X}_a$ of $\mathbb{X}$ with $\mathbb{X}_{a_{\min}} = \mathbb{X}$ for some $a_{\min} \in \mathcal{A}$ and $\cap_{a \in \mathcal{A}} \mathbb{X}_a = \{0\}$. Let $\mathcal{A}$ be partially ordered by

$$a \subset b \text{ iff } \mathbb{X}_b \subset \mathbb{X}_a.$$

Assume also that for $a, b \in \mathcal{A}$, the union of a and b, $a \cup b$, belongs to $\mathcal{A}$, and is defined so that

$$\mathbb{X}_a \cap \mathbb{X}_b = \mathbb{X}_{a \cup b}.$$

For the definition of $a \cup b$ in physical N-body Schrödinger operators, we refer to [20]. For each a, we denote $\mathbb{X}^a$ the orthogonal complement of $\mathbb{X}_a$ in $\mathbb{X}$. We write the corresponding orthogonal decomposition of coordinates x as

$$x = x^a + x_a.$$

With these notation, the N-body Schrödinger operators we are interested in are of the form

$$P(h) = -h^2\Delta + \sum_{a \in \mathcal{A}} V_a(x^a), \tag{2.4}$$

where $h > 0$ is a small parameter, Δ is the Laplacian on $(\mathbb{X}, q(\cdot))$. We assume that V_a satisfies

$$|\partial_y^\alpha V_a(y)| \leqslant C_\alpha r(y)\langle y\rangle^{-|\alpha|}, \quad y \in \mathbb{X}^a, \quad \forall \alpha \in \mathbb{N}^{d_a}. \tag{2.5}$$

Here $r(y) \to 0$ as $y \to \infty$.

For each $a \in \mathcal{A}$, we denote $\#a$ the number of clusters in a, $P^a(h)$ the cluster Hamiltonian

$$P^a(h) = -h^2\Delta^a + \sum_{b \subseteq a} V_b(x^b),$$

where Δ^a is the Laplacian in x^a-variables. Put

$$I_a(x) = \sum_{b \nsubseteq a} V_b(x^b), \quad P_a(h) = P^a(h) - h^2\Delta_a,$$

where Δ_a is the Laplacian in x_a-variables. Then one has: $P(h) = P_a(h) + I_a(x)$ for any cluster decomposition a. Let p^a denote the semi-classical symbol of $P^a(h)$. Assume that

$$\forall a, \quad p^a \text{ is non-trapping at the energy } E. \tag{2.6}$$

Under the assumptions (2.5) and (2.6), one can construct a conjugate operator, F, of $P(h)$ at E which is a self-adjoint semi-classical pseudo-differential operator satisfying

$$i\chi(P(h))[P(h), F]\chi(P(h)) \geqslant c_0 h \chi(P(h))^2, \quad h \in]0, 1], \tag{2.7}$$

where $c_0 > 0$ is independent of h and χ is a smooth real function on $\mathbb{R}$ supported sufficiently near E. See [19]. The non-trapping condition (2.6) is both necessary and sufficient for the following semi-classical resolvent estimate (2.8) [19].

Theorem 2.1. *Assume the conditions* (2.5) *and* (2.7). *The following estimates hold*:

(i) *For any* $l \in \mathbb{N}^*$, $s > l - \frac{1}{2}$, *there exists* $C > 0$ *such that*

$$\|\langle F\rangle^{-s}(R(E \pm i\kappa, h))^l \langle F\rangle^{-s}\| \leqslant Ch^{-l} \tag{2.8}$$

uniformly in $0 < \kappa < 1$.

(ii) *Let* $c_\pm \in \mathbb{R}$ *and let* $\chi_\pm$ *denote the characteristic functions of* $]-\infty, c_-[$ *and* $]c_+, +\infty[$, *respectively. For any* $l \in \mathbb{N}^*$, $r \geqslant 0$, $s > l - \frac{1}{2}$, *there exists* $C > 0$ *such that*

$$\|\langle F\rangle^{s-l}\chi_\mp(F)(R(E \pm i\kappa, h))^l \langle F\rangle^{-s}\| \leqslant Ch^{-l} \tag{2.9}$$

and

$$\|\langle F\rangle^{r}\chi_\mp(F)(R(E \pm i\kappa, h))^l \chi_\pm(F)\langle F\rangle^{r}\| \leqslant Ch^{-l}, \tag{2.10}$$

uniformly in $0 < \kappa < 1$. *Here* C *is some constant only depending on* c_0, l, s *and* r.

Remark 2.2. (a) The statement of (2.9) is slightly different from (1.6) of Theorem 1.1 in [19] which gives the smoothness of boundary values of the resolvent. Eq. (2.9) follows from an easy argument used, for example, in [20].

(b) When the conjugate operator F depends on an additional parameters, the Mourre method [13] allows to obtain uniform estimates on the resolvent as long as (2.7) holds uniformly w.r.t. these parameters. This idea was already exploited by Mourre in the proof of Theorem 1.2 in [14].

In the following, we shall use Mourre's idea to deduce from Theorem 2.1 the semi-classical resolvent estimates in Besov spaces.

Let $l^{2,\infty}$ denote the space of measurable functions $g(t)$ on $\mathbb{R}$ such that

$$\|g\|_{2,\infty} = \left\{\sum_{k\in\mathbb{Z}} |g|_k^2\right\}^{\frac{1}{2}},$$

where $|g|_k = \operatorname{ess\,sup}\{|g(t)|; k \leqslant t < k+1\}$, $k \in \mathbb{Z}$.

Corollary 2.3. *Let* $f_1, f_2 \in l^{2,\infty}$.

$$\|f_1(F)R(E \pm i\kappa, h)f_2(F)\| \leqslant Ch^{-1}\|f_1\|_{2,\infty}\|f_2\|_{2,\infty} \tag{2.11}$$

uniformly in $0 < \kappa < 1$.

Proof. We follow the Mourre's argument used in the proof of (III) of Theorem 1.2 in [14] in checking the h-dependence. See also [10,22] when $h = 1$. Let χ_n ($\chi_\pm$, resp.)

denote the characteristic function of $[n, n+1[$, $n \in \mathbb{Z}$, ($[0, +\infty[$, $]-\infty, 0[$, resp.). Then for $u, v \in L^2$,

$$
\begin{aligned}
&|\,(f_1(F)R(E \pm i\kappa, h) f_2(F)u,\ \ v)\,| \\
&\quad \leqslant \sum_{n,m\in\mathbb{Z}} |f_1|_n |f_2|_m \|\chi_n(F)v\|\ \|\chi_m(F)u\|\ \|\chi_n(F)R(E \pm i\kappa)\chi_m(F)\| \\
&\quad \leqslant \|u\|\ \|v\|\ \|f_1\|_{2,\infty}\|f_2\|_{2,\infty} \sup_{n,m\in\mathbb{Z}} \|\chi_n(F)R(E \pm i\kappa, h)\chi_m(F)\|.
\end{aligned}
$$

It remains to prove

$$
\sup_{n,m} \|\chi_n(F)R(E \pm i\kappa, h)\chi_m(F)\| \leqslant Ch^{-1} \tag{2.12}
$$

uniformly in $\kappa \in]0, 1]$. Note that $F - n$ is still a conjugate operator of $P(h)$ satisfying (2.7) with the same lower bound. The functional-analytic proof of (2.8) by Mourre's method gives that

$$
\|\chi_n(F)R(E \pm i\kappa, h)\chi_n(F)\| \leqslant Ch^{-1}
$$

uniformly in n and κ. Decompose $\chi_n(F)R(E + i\kappa, h)\chi_m(F)$ as

$$
\begin{aligned}
&\chi_n(F)R(E + i\kappa, h)\chi_m(F) \\
&\quad = \chi_n(F)\{\chi_-(F - m)R(E + i\kappa, h) + \chi_+(F - m)R(E - i\kappa, h) \\
&\qquad +2i\kappa\chi_+(F - m)R(E - i\kappa, h)R(E + i\kappa, h)\}\chi_m(F).
\end{aligned}
$$

The first two terms can be bounded by Ch^{-1} according to (2.9) with $l = 1$. For the third term, remark that

$$
\begin{aligned}
&2\kappa\|\chi_n(F)R(E - i\kappa, h)R(E + i\kappa, h)\chi_m(F)\| \\
&\quad \leqslant 4\|\chi_n(F)R(E + i\kappa, h)\chi_n(F)\|^{\frac{1}{2}}\|\chi_m(F)R(E + i\kappa, h)\chi_m(F)\|^{\frac{1}{2}} \\
&\quad \leqslant Ch^{-1}
\end{aligned}
$$

uniformly in n, m and κ. (2.12) is proved. $\square$

Let F_j, $j \in \mathbb{N}$, denote the spectral projector of F onto the set Ω_j, where $\Omega_j = \{\lambda \in \mathbb{R};\ 2^{j-1} \leqslant |\lambda| < 2^j\}$ for $j \geqslant 1$ and $\Omega_0 = \{\lambda \in \mathbb{R};\ |\lambda| < 1\}$. Introduce the abstract Besov spaces, B_s^F, defined in terms of the conjugate operator F

$$B_s^F = \left\{ u \in L^2; \sum_{k=0}^{\infty} 2^{ks} \|F_k u\| < \infty \right\}, \quad s \geqslant 0.$$

Its dual space $(B_s^F)^*$ w.r.t. the L^2-product is a Banach space with the norm

$$\|u\|_{(B_s^F)^*} = \sup_{j \in \mathbb{N}} 2^{-js} \|F_j u\|.$$

When F is replaced by $|x|$, one recovers the usual Besov spaces B_s and B_s^*.

Corollary 2.4. *Let* $s \geqslant \frac{1}{2}$. *One has*

$$\|R(E \pm i\kappa, h)\|_{\mathcal{L}(B_s^F, (B_s^F)^*)} \leqslant Ch^{-1} \tag{2.13}$$

uniformly in $0 < \kappa < 1$.

Proof. Let $f \in C_0^\infty(\mathbb{X})$. By Corollary 2.3, one has

$$\begin{aligned} &2^{-js} \|F_j R(E \pm i\kappa) f\| \\ &\quad \leqslant \sum_{k=0}^{\infty} 2^{-js} \|F_j R(E \pm i\kappa) F_k\| \|F_k f\| \\ &\quad \leqslant Ch^{-1} \sum_{k=0}^{\infty} 2^{-j(s-\frac{1}{2})} 2^{k/2} \|F_k f\| \leqslant Ch^{-1} \|f\|_{B_s^F} \end{aligned}$$

uniformly in h, κ and j. This proves (2.13). □

The semi-classical resolvent estimate in the usual Besov spaces can be easily deduced from Corollary 2.4.

Theorem 2.5. *Let* $s \geqslant \frac{1}{2}$. *Under the assumptions* (2.5) *and* (2.6), *one has*

$$\|R(E \pm i\kappa, h)\|_{\mathcal{L}(B_s, B_s^*)} \leqslant Ch^{-1} \tag{2.14}$$

uniformly in $0 < \kappa < 1$.

Proof. Let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(t) = 1$ for t near E. $(1 - \chi(P(h))^2) R(E \pm i\kappa, h)$ is uniformly bounded in $\mathcal{L}(L^2, L^2)$, therefore also in $\mathcal{L}(B_s, B_s^*)$. Let F be a semi-classical pseudo-differential operator with Weyl symbol $x \cdot \xi + r(x, \xi)$, where r is a bounded symbol (cf. [19]). We can show that for $s \geqslant 0$,

$$\|\langle F\rangle^s \chi(P(h))\langle x\rangle^{-s}\| \leqslant C \tag{2.15}$$

uniformly in h. An argument of interpolation (cf. [1]) gives then

$$\|\chi(P(h))\|_{\mathcal{L}(B_s, B_s^F)} \leqslant C$$

uniformly in h. By duality, the same is true for $\chi(P(h))$ as operator from $(B_s^F)^*$ to B_s^*. It follows that

$$\|\chi(P(h))^2 R(E \pm i\kappa, h)\|_{\mathcal{L}(B_s, B_s^*)} \leqslant Ch^{-1},$$

which completes the proof of (2.14). $\square$

The regularity on potentials is only needed to make use of theory of pseudo-differential operators. If we make the assumption

$$2E - 2V(x) - x\nabla V(x) \geqslant c_0 > 0, \quad \forall x, \tag{2.16}$$

which is stronger than (2.6), the condition (2.5) can be considerably weakened. See [13].

Remark on Morrey–Campanato estimate. In [15], the authors mentioned that it is interesting in itself to study Morrey–Campanato estimates for Schödinger operators. We indicate here how this kind of estimates can be deduced from Besov space estimates.

Denote the Morrey–Companato norm

$$|||u|||^2 = \sup_{R>0} \frac{1}{R} \int_{|x|<R} |u|^2\, dx$$

and $N(f)$ the dual norm

$$N(f) = \sum_{j\in\mathbb{Z}} \left(2^{j+1} \int_{C(j)} |f|^2\, dx\right)^{\frac{1}{2}},$$

where $C(j) = \{x \in \mathbb{R}^d; 2^j \leqslant |x| \leqslant 2^{j+1}\}$. When $h > 0$ is fixed, assume that $V_a(y)$, $y \cdot \nabla_y V_a(y)$ are relatively compact perturbation of $-\Delta_y$ and $(y \cdot \nabla_y)^2 V_a(y)$ is relatively form-bounded w.r.t. $-\Delta_y$. Then for E outside the thresholds and eigenvalues of $P = -\Delta + \sum_{a\in\mathcal{A}} V_a(x^a)$, one has for $d \geqslant 3$

$$|||(P - (E \pm i\kappa))^{-1} u||| \leqslant CN(u) \tag{2.17}$$

uniformly in κ. Indeed, for such E, the Mourre estimate holds and it follows that $(P-(E\pm i\kappa))^{-1}$ is bounded in $\mathcal{L}(B_{\frac{1}{2}}, B^*_{\frac{1}{2}})$. Eq. (2.17) then follows by noticing that $\|(-\Delta+1)^{-1/2} f\|_{B_{\frac{1}{2}}} \leqslant CN(f)$ if $d \geqslant 3$ (see more details below). In the semi-classical case, the same argument gives rise to an additional loss on the power of h^{-1} due to the singularity of $|x|^{-\frac{1}{2}}$ at the origin. The following parameter-dependent result may be useful to the study of high-frequency limit of Helmholtz equation.

Proposition 2.6. *Let $P_\varepsilon = -\Delta + V_\varepsilon(x)$ be a Schrödinger operator with potential depending on a small parameter $\varepsilon \in]0,1]$. Assume that $V_\varepsilon(x)$ and $(x\cdot\nabla_x)^j V_\varepsilon(x)$, $j=1,2$, are uniformly bounded in ε and $x \in \mathbb{R}^d$ with $d \geqslant 3$. Let $E>0$. Assume further that there exist $c_0>0$, $\delta>0$ such that*

$$\chi(P_\varepsilon) i[P_\varepsilon, F_0]\chi(P_\varepsilon) \geqslant c_0 \chi(P_\varepsilon)^2 \tag{2.18}$$

for some $\chi \in C_0^\infty(]E-\delta, E+\delta[;\mathbb{R}_+)$ with $\chi(s)=1$ for $s\in[E-\delta/2, E+\delta/2]$, and for all $\varepsilon \in]0,1]$. Here $F_0 = (x\cdot D_x + D_x \cdot x)/2$. Then one has

$$|||(P_\varepsilon - (E\pm i\kappa))^{-1}u||| \leqslant CN(u) \tag{2.19}$$

for all $u \in L^2_{\mathrm{loc}}$ with $N(u)<\infty$, uniformly in ε, κ.

Remark 2.7. (a) Eq. (2.19) is proved in [15] for $V_\varepsilon = E - n^2(\varepsilon x)$ under the condition

$$2\sum_{j\in\mathbb{Z}} \sup_{2^j<|x|\leqslant 2^{j+1}} \frac{(x\cdot\nabla_x n^2(x))_-}{n^2(x)} < 1. \tag{2.20}$$

(b) For $V_\varepsilon(x) = V(\varepsilon x)$, if V satisfies (2.16), (2.18) is true. Note that the condition (2.16) is satisfied if $n^2 = E - V$ verifies

$$\frac{1}{2}\sum_{j\in\mathbb{Z}} \sup_{2^j<|x|\leqslant 2^{j+1}} \frac{(x\cdot\nabla_x n^2(x))_-}{n^2(x)} < 1.$$

and $n^2(x) \geqslant n_0^2 > 0$. Thus, Proposition 2.6 may be regarded as an alternative approach to prove the Morrey–Campanato estimate of Perthame and Vega [15].

Proof. By the Mourre method in Besov spaces, under the condition (2.18), one can show that

$$||(P_\varepsilon - (E\pm i\kappa))^{-1}v||_{B^*_{\frac{1}{2}}} \leqslant C\|v\|_{B_{\frac{1}{2}}}, \quad \forall v \in B_{\frac{1}{2}} \tag{2.21}$$

uniformly in ε, κ.

Recall the Hardy inequality for $d \geqslant 3$

$$\||x|^{-1} f\|^2_{L^2(\mathbb{R}^d)} \leqslant \frac{4}{(d-2)^2} \|\nabla f\|^2_{L^2(\mathbb{R}^d)}, \quad f \in C_0^\infty(\mathbb{R}^d).$$

It follows that $|x|^{-1}(1-\Delta)^{-1/2}$ and $(1-\Delta)^{-1/2}|x|^{-1}$ are bounded as operators on L^2. By a complex interpolation, we obtain that for any $0 \leqslant s \leqslant 1$,

$$|x|^{-s}(1-\Delta)^{-s/2} \quad \text{and} \quad (1-\Delta)^{-s/2}|x|^{-s} \in \mathcal{L}(L^2). \tag{2.22}$$

Let χ_1 be a cut-off on $\mathbb{R}^d$ with $\chi_1(x) = 1$ for $|x| \leqslant 1$, and 0 for $|x| \geqslant 2$. Set $\chi_2 = 1 - \chi_1$. On supp χ_2, $B_{\frac{1}{2}}$ (resp., $B^*_{\frac{1}{2}}$) norm is equivalent with $N(\cdot)$ (resp., $|||\cdot|||$). Since $(-\Delta+1)^{-\frac{1}{2}}$ is bounded from $B_{\frac{1}{2}}$ to $B_{\frac{1}{2}}$ and from $B^*_{\frac{1}{2}}$ to $B^*_{\frac{1}{2}}$, splitting u as $u = \chi_1 u + \chi_2 u$ and applying (2.22) to $\chi_1 u$ with appropriate $\frac{1}{2} \leqslant s \leqslant 1$, one has

$$\|(-\Delta+1)^{-\frac{1}{2}} u\|_{B_{\frac{1}{2}}} \leqslant CN(u), \tag{2.23}$$

$$|||(-\Delta+1)^{-\frac{1}{2}} u||| \leqslant C\|u\|_{B^*_{\frac{1}{2}}}. \tag{2.24}$$

Let χ be the same cut-off function as that in Proposition 2.6. Then for all $u \in C_0^\infty(\mathbb{R}^d)$, one has

$$|||(1-\chi(P_\varepsilon)^2)(P_\varepsilon - (E \pm i\kappa))^{-1} u||| \leqslant C\|(-\Delta+1)^{-\frac{1}{2}} u\|$$

uniformly in ε, κ. By (2.23),

$$|||(1-\chi(P_\varepsilon)^2)(P_\varepsilon - (E \pm i\kappa))^{-1} u||| \leqslant C' N(u).$$

On the other hand, by (2.21), (2.23), (2.24) and the argument used above, one has

$$\begin{aligned} |||\chi(P_\varepsilon)^2(P_\varepsilon - (E \pm i\kappa))^{-1} u||| &\leqslant C\|(P_\varepsilon - (E \pm i\kappa))^{-1}\chi(P_\varepsilon) u\|_{B^*_{\frac{1}{2}}} \\ &\leqslant C_1\|\chi(P_\varepsilon) u\|_{B_{\frac{1}{2}}} \leqslant C_2 N(u) \end{aligned}$$

for all $u \in C_0^\infty$, uniformly in ε, κ. Combining the above two estimates, we obtain the desired estimate for $u \in C_0^\infty$. An argument of density completes the proof of Proposition 2.6. □

3. A priori estimates for the Helmholtz equation

In [5], the authors divided the Helmholtz equation with a source, which concentrates near a p-dimensional submanifold Γ, into three regimes: resonant regime, propagative regime and characteristic regime, according to the relations between $|\nabla^\tau \phi(y)|$ and $n(y)$ for $y \in \Gamma$. In the following, we study the Helmholtz equation (1.1) for $\Gamma = \mathbb{R}^p \subset \mathbb{R}^d$ in the propagative regime where

$$q = \frac{3+d+p}{2} \quad \text{and} \quad |\nabla\phi(x_1)|^2 < n^2(x_1, 0), \quad \forall x_1 \in \mathbb{R}^p. \tag{3.1}$$

The resonant regime is easier, while the characteristic regime is the most difficult case, it is not yet rigourously studied even for constant refraction index.

Now let us give an a priori estimate for the solutions to (1.1). With the notation of Section 2, one has

$$u_\varepsilon = -\varepsilon^2 R(E + i\kappa) S_\varepsilon, \quad \kappa = \kappa(\varepsilon) \equiv \varepsilon\alpha_\varepsilon.$$

Due to the particular form of S_ε, the result of Section 2 cannot be directly applied here. However, we shall show how the idea of the proof allows to give a uniform estimate on u_ε. For $x \in \mathbb{R}^d$, set $x = (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{d-p}$. Eq. (2.1) can be rewritten as

$$\big((\varepsilon D_1 + \nabla_1\phi(x_1))^2 - \Delta_2 + V(x_1, \varepsilon x_2) - (E + i\kappa)\big) w_\varepsilon = \varepsilon^{\frac{1}{2}} T_\varepsilon(x), \quad \kappa = \kappa(\varepsilon), \tag{3.2}$$

where $\nabla_j = \nabla_{x_j}$, $D_j = -i\nabla_j$, Δ_j is Laplacian in x_j, $j = 1, 2$,

$$w_\varepsilon = \varepsilon^{(d-p)/2} e^{-i\phi(x_1)/\varepsilon} u_\varepsilon(x_1, \varepsilon x_2)$$

and

$$T_\varepsilon(x) = \int_{\mathbb{R}^p} e^{i\phi(x_1, y_1, \varepsilon)} A(x_1 - \varepsilon y_1) S(y_1, x_2)\, dy_1 \tag{3.3}$$

with

$$\phi(x_1, y_1, \varepsilon) = \int_0^1 y_1 \cdot \nabla_1\phi(x_1 - \varepsilon\tau y_1)\, d\tau.$$

With the above formulation, to use the Mourre approach to get a uniform estimate on the solutions of (1.1), we need the following assumptions:

(A.1) $V(x)$ and $(x \cdot \nabla)^j V(x)$, $j = 1, 2$, are bounded on $\mathbb{R}^d$.

(A.2)

$$E - |\nabla_1 \phi(x_1)|^2 - V(x) - x \cdot \nabla V(x) \geqslant \delta_0 > 0, \quad \forall x \in \mathbb{R}^d \tag{3.4}$$

for some $\delta_0 > 0$. When $n(x)$ is a constant, this condition is the same as the assumption (H3) of Castella et al. [5], which is characteristic of the uniform propagative regime. We need also the following assumption on the smallness of the second derivatives of ϕ

(A.3)

$$|(\partial^2 \phi)(x_1) \cdot x_1| \leqslant \varepsilon_0 \tag{3.5}$$

for some $\varepsilon_0 > 0$ with $2\varepsilon_0 \|\nabla_1 \phi\|_\infty + \varepsilon_0^2 < \delta_0$. Here $\partial^2 \phi$ denotes the Hessian of ϕ. Eq. (3.5) is satisfied when the phase ϕ is linear $\phi(x_1) = v \cdot x_1$, $v \in \mathbb{R}^p$, or when $\phi(x_1) = a|x_1|(1 - \chi(\eta|x_1|)) + \chi(\eta|x_1|)$, where $a \in \mathbb{R}$ is arbitrary, $\eta > 0$ is small enough and $\chi \in C_0^\infty$ with $\chi(|x_1|) = 1$ for $|x_1|$ small. In the second case, one can use $(\partial^2 |x_1|) \cdot x_1 = 0$, $x_1 \neq 0$, to verify the condition (3.5) for $\eta > 0$ small. Remark that the propagative regime assumption implies that $|\nabla_1 \phi(x_1)|$ is bounded.

Theorem 3.1. *Under the above assumptions, one has for* $s > \frac{1}{2}$

$$\|u_\varepsilon\|_{B^*_{\frac{1}{2}}} \leqslant C \|A\|_{H^{1,s}(\mathbb{R}^p)} \|\langle x_1 \rangle \langle x \rangle^s S\|_{L^1(\mathbb{R}^p_{x_1}; H^1(\mathbb{R}^{d-p}_{x_2}))} \tag{3.6}$$

uniformly in $\varepsilon \in]0, 1]$. *Here* $H^{1,s}(\mathbb{R}^d)$ *is the weighted Sobolev space of order* 1: $H^{1,s}(\mathbb{R}^d;\ dx) = H^1(\mathbb{R}^d;\ \langle x \rangle^{2s}\, dx)$.

Proof. Let $Q(\varepsilon) = (\varepsilon D_1 + \nabla_1 \phi(x_1))^2 - \Delta_2 + V(x_1, \varepsilon x_2)$ and $F_0 = (x \cdot D_x + D_x \cdot x)/2$. Let $q(x, \xi) = (\xi_1 + \nabla_1 \phi(x_1))^2 + \xi_2^2 + V(x)$. The Poisson bracket between q and $x \cdot \xi$ can be estimated as

$$\begin{aligned}
\{q(x, \xi), x \cdot \xi\} &= 2\xi^2 + 2\xi_1 \cdot \nabla_1 \phi - x \cdot \nabla V - 2x_1 \cdot \partial^2 \phi(x_1) \cdot (\nabla_1 \phi + \xi_1) \\
&= q(x, \xi) + \xi^2 - |\nabla_1 \phi|^2 - V - x \cdot \nabla V - 2x_1 \cdot \partial^2 \phi(x_1) \cdot (\nabla_1 \phi + \xi_1) \\
&\geqslant q(x, \xi) - E + \delta_0 - 2\varepsilon_0 \sup |\nabla_1 \phi(x_1)| - \varepsilon_0^2 \\
&\geqslant q(x, \xi) - E + \delta_0', \quad \delta_0' > 0.
\end{aligned} \tag{3.7}$$

In the third inequality of the above formula, we used

$$\xi^2 - 2x_1 \cdot \partial^2 \phi(x_1) \cdot \xi_1 \geqslant \xi^2 - \xi_1^2 - \sup |x_1 \cdot \partial^2 \phi(x_1)|^2 \geqslant -\varepsilon_0^2.$$

Using the estimate (3.7), we obtain for $\varepsilon > 0$ small enough

$$i[Q(\varepsilon), F_0] \geqslant Q(\varepsilon) - E + \delta_0'',$$

where $\delta_0'' > 0$. It follows that for $\chi \in C_0^\infty(]E - \delta_0''/2, E + \delta_0''/2[)$ with $\chi(t) = 1$ in a neighborhood of E

$$i\chi(Q(\varepsilon))[Q(\varepsilon), F_0]\chi(Q(\varepsilon)) \geqslant c_0\chi(Q(\varepsilon))^2, \quad c_0 > 0 \tag{3.8}$$

uniformly in ε. Repeating the argument for the proof of Corollary 2.4, we obtain

$$\|(Q(\varepsilon) - E \mp i\kappa)^{-1}\|_{\mathcal{L}(B_s^{F_0}, (B_s^{F_0})^*)} \leqslant C, \quad s \geqslant \tfrac{1}{2} \tag{3.9}$$

uniformly in ε and κ, where $B_{\frac{1}{2}}^{F_0}$ is the abstract Besov space defined in Section 2 with F there replaced by F_0. Since $\langle F_0\rangle^2(-\varepsilon^2\Delta_1 - \Delta_2 + 1)^{-1}\langle(\varepsilon^{-1}x_1, x_2)\rangle^{-2}$ is uniformly bounded on L^2, we can show as before that $\chi(Q(\varepsilon))$ is uniformly bounded as operator from $B_{\frac{1}{2}}(\varepsilon)$ to $B_{\frac{1}{2}}^{F_0}$, and by the duality, from $(B_{\frac{1}{2}}^{F_0})^*$ to $B_{\frac{1}{2}}(\varepsilon)^*$, where $B_s(\varepsilon)$ is the Besov space constructed with the Euclidean norm $|x|$ replaced by the scaled one $|x|_\varepsilon = (\varepsilon^{-2}|x_1|^2 + |x_2|^2)^{\frac{1}{2}}$ in the decomposition of $\mathbb{R}^d$. It follows that

$$\|w_\varepsilon\|_{B_{\frac{1}{2}}(\varepsilon)^*} \leqslant C\varepsilon^{\frac{1}{2}}\|T_\varepsilon\|_{B_{\frac{1}{2}}^{F_0}}. \tag{3.10}$$

For $0 < \varepsilon < 1$, one has

$$\begin{aligned}
\|w_\varepsilon\|_{B_{\frac{1}{2}}(\varepsilon)^*} &= \sup_{R>1} \frac{1}{R^{\frac{1}{2}}}\left(\int_{|x|_\varepsilon<R} |u_\varepsilon(x_1, \varepsilon x_2)|^2 \varepsilon^{d-p}\,dx\right)^{\frac{1}{2}} \\
&= \varepsilon^{\frac{1}{2}} \sup_{R>1} \frac{1}{(\varepsilon R)^{\frac{1}{2}}}\left(\int_{|x|<\varepsilon R} |u_\varepsilon(x)|^2\,dx\right)^{\frac{1}{2}} \\
&\geqslant \varepsilon^{\frac{1}{2}} \sup_{R'>1} \frac{1}{R'^{\frac{1}{2}}}\left(\int_{|x|<R'} |u_\varepsilon(x)|^2\,dx\right)^{\frac{1}{2}} = \varepsilon^{\frac{1}{2}}\|u_\varepsilon\|_{B_{\frac{1}{2}}^*}.
\end{aligned}$$

It follows that

$$\|u_\varepsilon\|_{B_{\frac{1}{2}}^*} \leqslant C\|T_\varepsilon\|_{B_{\frac{1}{2}}^{F_0}}. \tag{3.11}$$

On the other hand, using Minkowski's inequality, we find

$$\|T_\varepsilon\| \leqslant \left\{ \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^p} |A(x_1 - \varepsilon y_1) S(y_1, x_2)| \, dy_1 \right)^2 dx_1 \, dx_2 \right\}^{\frac{1}{2}}$$

$$\leqslant \int_{\mathbb{R}^p} \left\{ \int_{\mathbb{R}^d} |A(x_1 - \varepsilon y_1) S(y_1, x_2)|^2 \, dx_1 \, dx_2 \right\}^{\frac{1}{2}} dy_1$$

$$= \|A\|_{L^2(\mathbb{R}^p)} \|S\|_{L^1(\mathbb{R}^p_{x_1}; L^2(\mathbb{R}^{d-p}_{x_2}))}.$$

Since $|\partial_{x_1}^{\alpha_1} \phi(x_1, y_1, \varepsilon)| = O(|y_1|)$ for $|\alpha_1| \leqslant 1$, we obtain by the same argument that for $|\alpha| \leqslant 1$,

$$\|\langle x \rangle^s \partial_x^\alpha T_\varepsilon\| \leqslant C(\phi) \|A\|_{H^{1,s}(\mathbb{R}^p)} \|\langle x_1 \rangle \langle x \rangle^s S\|_{L^1(\mathbb{R}^p_{x_1}; H^1(\mathbb{R}^{d-p}_{x_2}))}$$

for any $0 \leqslant s \leqslant 1$. Since $\|T_\varepsilon\|_{B^{F_0}_{\frac{1}{2}}} \leqslant \|\langle F_0 \rangle^s T_\varepsilon\| \leqslant C_s \|T_\varepsilon\|_{H^{1,s}}$ for any $\frac{1}{2} < s \leqslant 1$, $\|u_\varepsilon\|_{B^*_{\frac{1}{2}}}$ is uniformly controlled by the right-hand side of (3.6) for any $s > \frac{1}{2}$. $\square$

Remark 3.2. (a) For constant refraction index $n(x) = n_0$, it is proved in [5] that

$$\|u_\varepsilon\|_{B^*_{\frac{p+1}{2}}} \leqslant C$$

uniformly in ε.

(b) The conditions (A.2) and (A.3) ensure the validity of (3.7), which implies that the classical Hamiltonian $q(x, \xi)$ is non-trapping at E. By the results of Wang [18,19], a non-trapping condition for q is necessary to obtain a uniform resolvent estimate for $Q(\varepsilon)$ in $\mathcal{L}(B_{\frac{1}{2}}, B^*_{\frac{1}{2}})$. Since the right-hand side T_ε has a particular structure, one may hope to improve these conditions. Under the conditions (A.1) and (A.5) below with $z_1 = 0$, we can show that u_ε is uniformly bounded in $(B^{F_0}_{\frac{1}{2}})^*$. However, this uniform estimate seems too weak to construct a limiting semi-classical measure, due to the "loss of half derivative" in $(B^{F_0}_{\frac{1}{2}})^*$.

Theorem 3.1 allows us to show that $\{u_\varepsilon\}$ admits a subsequence $\{u_{\varepsilon_k}\}$ such that the Wigner transform of u_{ε_k} is uniformly bounded in the dual space of some Banach space, X_λ, for any $\lambda > 1$ which will be defined in Section 4. Therefore, the corresponding limiting semi-classical measure exists. Following the idea of Castella et al. [5], to compute the limiting source term in the Liouville equation, we need an a priori estimate for

$$w^\varepsilon_{z_1}(y) = \varepsilon^{\frac{(d-p-1)}{2}} u_\varepsilon(z_1 + \varepsilon y_1, \varepsilon y_2) e^{-\frac{i\phi(z_1)}{\varepsilon}}, \tag{3.12}$$

where $z_1 \in \mathbb{R}^p$ is a parameter. Set

$$V_{z_1}(y) = V(z_1 + y_1, y_2), \quad A_{z_1}(y_1) = A(z_1 + y_1), \quad \phi_{z_1}(y_1) = \phi(z_1 + y_1)$$

for $y = (y_1, y_2) \in \mathbb{R}^p \times \mathbb{R}^{d-p}$. Then from (1.1), $w^\varepsilon_{z_1}(y)$ satisfies

$$(-\triangle_y + V_{z_1}(\varepsilon y) - E - i\kappa) w^\varepsilon_{z_1}(y) = W^\varepsilon_{z_1}(y), \tag{3.13}$$

where $W^\varepsilon_{z_1}$ is defined in (1.10). To show that $w^\varepsilon_{z_1}(y)$ is uniformly bounded in an appropriate space, we need the uniform propagative regime assumption (see also [5]): for some $\delta_0 > 0$

(A.4)

$$E - |\nabla \phi(y_1)|^2 - V(y) \geqslant \delta_0 \tag{3.14}$$

and a reinforced non-trapping condition: for any $z_1 \in \mathbb{R}^d$ there exists $\delta_1 = \delta_1(z_1) > 0$ such that

(A.5)

$$E - V_{z_1}(y) - \tfrac{1}{2}(y \cdot \nabla_y) V_{z_1}(y) \geqslant \delta_1 \tag{3.15}$$

for all y. In the following, the estimates may depend on z_1, but they are locally uniform for $z_1 \in \mathbb{R}^p$. For a symbol b on $\mathbb{R}^{2d}$, we denote $b(\varepsilon y, D)$ the pseudo-differential operator defined by

$$b(\varepsilon y, D)u(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(y-x)\cdot\xi} b(\varepsilon y, \xi) u(x)\, dx\, d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d).$$

The following parameter-dependent microlocal resolvent estimates play an important role in this work. See also [18,20].

Theorem 3.3. *Let* $P_\varepsilon = -\triangle + V_{z_1}(\varepsilon y)$ *and* $R_\varepsilon(\lambda) = (P_\varepsilon - \lambda)^{-1}$.

(a) *Under the conditions* (A.1) *and* (A.5), *one has*

$$\| R_\varepsilon(E \pm i\kappa) \|_{\mathcal{L}(B_{\frac{1}{2}}, B^*_{\frac{1}{2}})} \leqslant C \tag{3.16}$$

uniformly in ε *and* κ.

(b) *Let* $b_\pm(y,\xi)$ *be bounded symbols on* $\mathbb{R}^{2d}$ *with*

$$\operatorname{supp} b_\pm \subset \{y;\, |y| \leqslant R\} \cup \{(y,\xi),\, \pm y\cdot\xi > -(1-\eta)|y||\xi|\}$$

for some $R > 0$ *and* $2 > \eta > 0$. *Under the conditions* (1.3), (1.4) *for* $\rho_0 > 0$ *and* (A.5), *one has for any* $s > \frac{1}{2}$,

$$\|\langle y\rangle^{-s} R_\varepsilon(E \pm i\kappa) b_\pm(\varepsilon y, D)\langle y\rangle^{s-1}\| \leqslant C \tag{3.17}$$

uniformly in ε *and* κ.

(c) *Under the assumptions of* (b), *suppose further that* $b_\pm(y,\xi)$ *are bounded symbols satisfying*

$$\operatorname{supp} b_\pm \subset \{y;\, |y| \leqslant R\} \cup \{(y,\xi),\, \pm y\cdot\xi > \pm\eta_\pm|y||\xi|\}$$

for $-1 < \eta_- < \eta_+ < 1$. *Then, for any* $s, s' \geqslant 0$,

$$\|\langle y\rangle^{s} b_\mp(\varepsilon y, D) R_\varepsilon(E \pm i\kappa) b_\pm(\varepsilon y, D)\langle y\rangle^{s'}\| \leqslant C \tag{3.18}$$

uniformly in ε *and* κ.

Proof. (a) Under the condition (3.15), one can show that $F_0 = (y\cdot D_y + D_y\cdot y)/2$ is a conjugate operator of P_ε at the energy E: there exist $c_0 > 0$ and $\eta > 0$ such that for any $\chi \in C_0^\infty(]E-\eta, E+\eta[;\ \mathbb{R}_+)$, one has

$$i\chi(P_\varepsilon)[P_\varepsilon, F_0]\chi(P_\varepsilon) \geqslant c_0\chi(P_\varepsilon)^2,$$

uniformly in ε. Then, the Mourre method depending on a parameter given in Section 2 implies that

$$\|(P_\varepsilon - E \mp i\kappa)^{-1}\|_{\mathcal{L}(B_s^{F_0}, (B_s^{F_0})^*)} \leqslant C, \quad s \geqslant \tfrac{1}{2}. \tag{3.19}$$

Eq. (3.16) follows from the argument used in Theorem 2.5.

(b) and (c) Eqs. (3.17) and (3.18) can be proved by the method of outgoing and incoming parametrices of Wang [18]. Note that P_ε is unitarily equivalent with $-\varepsilon^2\triangle_x + V_{z_1}(x)$. The parametrices needed for the representation of the resolvent $R_\varepsilon(E \pm i\kappa)$ can be obtained from those of Wang [18] by a unitary transformation. Note also that we use here the uniform estimate (3.16) to replace the semi-classical resolvent estimate of Robert and Tamura [16] in the form

$$\|\langle x\rangle^{-s}(-h^2\Delta + V(x) - E - i\kappa)^{-1}\langle x\rangle^{-s}\| \leqslant Ch^{-1}, \quad s > \tfrac{1}{2}$$

used in [18], so that we can obtain a uniform upper-bound in (3.17) and (3.18). The details are omitted here. $\square$

Theorem 3.4. *Under the assumptions* (1.3), (1.4), (A.4) *and* (A.5), *one has for* $s > p/2 + 1$,

$$\|w_{z_1}^{\varepsilon}\|_{L^{2,-s}} \leqslant C \tag{3.20}$$

uniformly in $\varepsilon, \kappa > 0$.

Proof. By change of variables, the right-hand side of (3.13) can be estimated by

$$|W_{z_1}^{\varepsilon}(y_1, y_2)| \leqslant \|A\|_{L^\infty} \int_{\mathbb{R}^p} |S(X_1, y_2)|\, dX_1. \tag{3.21}$$

Take $\delta > 0$ small enough, $\chi_1 \in C_0^\infty(\mathbb{R})$ with $0 \leqslant \chi_1 \leqslant 1$, $\operatorname{supp} \chi_1 \subset]E - \delta, E + \delta[$ and $\chi_1(t) = 1$ for t near E. Set $\chi_2 = 1 - \chi_1$ and

$$w_j^{\varepsilon} = R_{\varepsilon}(E + i\kappa)\chi_j(P_{\varepsilon})(W_{z_1}^{\varepsilon}), \quad j = 1, 2.$$

Since $\chi_2 = 0$ near E, one can check that $R_{\varepsilon}(E + i\kappa)\chi_2(P_{\varepsilon})$ is uniformly (in κ and ε) continuous in any weighted L^2-space on $\mathbb{R}^d$. By a method of interpolation (see [9, Theorem 14.1.1]), it is uniformly continuous in any Besov space. Therefore, w_2^{ε} can be estimated as

$$\begin{aligned}
\|w_2^{\varepsilon}\|_{B^*_{\frac{p}{2}}(\mathbb{R}^d)} &\leqslant C\|W_{z_1}^{\varepsilon}\|_{B^*_{\frac{p}{2}}(\mathbb{R}^d)} \\
&\leqslant C \sup_{R>1} \frac{1}{R^{\frac{p}{2}}} \left(\int_{|y_2| \leqslant R} \int_{|y_1| \leqslant R} |W_{z_1}^{\varepsilon}(y_1, y_2)|^2\, dy_1\, dy_2 \right)^{\frac{1}{2}} \\
&\leqslant C\|A\|_{L^\infty} \|S\|_{L^2(R_{x_2}^{d-p}; L^1(R_{x_1}^p))}.
\end{aligned} \tag{3.22}$$

To study w_1^{ε}, we introduce a partition of unity $\psi_1(\xi_1) + \psi_2(\xi_1) = 1$, where $\psi_j \in C^\infty, 0 \leqslant \psi_j(\xi_1) \leqslant 1$, $\operatorname{supp} \psi_2 \subset \{|\xi_1| < 2\delta\}$ and $\psi_2 = 1$ on $\{|\xi_1| < \delta\}$. Then

$$w_1^{\varepsilon} = R_{\varepsilon}(E + i\kappa)\chi_1(P_{\varepsilon}) \left(\Psi_1(z_1, \varepsilon) W_{z_1}^{\varepsilon} + \Psi_2(z_1, \varepsilon) W_{z_1}^{\varepsilon} \right) \triangleq w_{1,1}^{\varepsilon} + w_{1,2}^{\varepsilon}, \tag{3.23}$$

where $\Psi_j(z_1, \varepsilon)$ is a pseudo-differential operator so that for any $g \in \mathcal{S}(\mathbb{R}^p)$,

$$\Psi_j(z_1, \varepsilon) g(x_1) = \frac{1}{(2\pi)^p} \int \int_{\mathbb{R}^p \times \mathbb{R}^p} e^{i(x_1 - y_1)\xi_1} \psi_j(\xi_1 - \nabla \phi_{z_1}(\varepsilon y_1)) g(y_1)\, dy_1\, d\xi_1.$$

In the subsequence, we will derive the uniform estimate for w_1^ε. For clearness, we divide the proof into the following two steps.

Step 1: The estimate of $w_{1,1}^\varepsilon$.

In this step, we show that the method of non-stationary phase can be applied to estimate $w_{1,1}^\varepsilon$. Firstly for any positive number $0 < \alpha < 1$, we denote

$$\begin{aligned} v^\varepsilon &= \Psi_1(z_1,\varepsilon)W_{z_1}^\varepsilon(x_1,y_2) \\ &= \int\int_{\mathbb{R}^p\times\mathbb{R}^p} e^{i(x_1-y_1)\xi_1}\psi_1(\xi_1-\nabla\phi_{z_1}(\varepsilon y_1))\int_{\mathbb{R}^p} e^{i\,\frac{\phi_{z_1}(\varepsilon(y_1-X_1))-\phi_{z_1}(0)}{\varepsilon}} \\ &\quad\times(\psi_1(\varepsilon^\alpha X_1)+\psi_2(\varepsilon^\alpha X_1))A_{z_1}(\varepsilon(y_1-X_1))S(X_1,y_2)\,dX_1\,dy_1\,d\xi_1 \\ &\triangleq v_1^\varepsilon+v_2^\varepsilon. \end{aligned} \tag{3.24}$$

Then for $1\leqslant j\leqslant p$, we have

$$\begin{aligned} &-ix_jv_1^\varepsilon(x_1,y_2) \\ &= \int\int_{\mathbb{R}^p\times\mathbb{R}^p} e^{i(x_1-y_1)\xi_1}\big(-iy_j\psi_1(\xi_1-\nabla\phi_{z_1}(\varepsilon y_1))+\partial_j\psi_1(\xi_1-\nabla\phi_{z_1}(\varepsilon y_1))\big) \\ &\quad\times\int_{\mathbb{R}^p} e^{i\,\frac{\phi_{z_1}(\varepsilon(y_1-X_1))-\phi_{z_1}(0)}{\varepsilon}}\psi_1(\varepsilon^\alpha X_1)A_{z_1}(\varepsilon(y_1-X_1))S(X_1,y_2)\,dX_1\,dy_1\,d\xi_1 \\ &\triangleq k_1^\varepsilon+k_2^\varepsilon. \end{aligned} \tag{3.25}$$

Note that on the support of $\psi_1(\varepsilon^\alpha\cdot)$, $\varepsilon|X_1|\leqslant C\varepsilon^{1-\alpha}$, then for ε sufficiently small, we have

$$|\nabla\phi_{z_1}(\varepsilon(y_1-X_1))-\nabla\phi_{z_1}(\varepsilon y_1)|\leqslant C(\varepsilon)\to 0 \quad \text{as } \varepsilon\to 0,$$

on the support of $\psi_1(\varepsilon^\alpha X_1)$, while on the support of $\psi_1(\cdot-\nabla\phi_{z_1}(\varepsilon y_1))$,

$$|\xi_1-\nabla\phi_{z_1}(\varepsilon y_1)|\geqslant\delta,$$

therefore on the support of $\psi_1(\xi_1-\nabla\phi_{z_1}(\varepsilon y_1))\psi_1(\varepsilon^\alpha X_1)$, we obtain

$$|\xi_1-\nabla\phi_{z_1}(\varepsilon(y_1-X_1))|\geqslant\frac{\delta}{2}. \tag{3.26}$$

We now denote

$$\Phi_1(y_1, \xi_1) = -y_1\xi_1 + \frac{\phi_{z_1}(\varepsilon(y_1 - X_1)) - \phi_{z_1}(0)}{\varepsilon},$$
$$B_1(D_{y_1}) = \frac{(\nabla\phi_{z_1}(\varepsilon(y_1 - X_1)) - \xi_1) \cdot D_{y_1}}{|\nabla\phi_{z_1}(\varepsilon(y_1 - X_1)) - \xi_1|^2}$$

with $D = \frac{1}{i}\partial$. Observing that

$$B_1(D_{y_1})e^{i\Phi_1(y_1,\xi_1)} = e^{i\Phi_1(y_1,\xi_1)},$$

then by (3.26) and integration by parts, we obtain

$$\begin{aligned} k_1^\varepsilon &= \int\int_{\mathbb{R}^p\times\mathbb{R}^p} e^{ix_1\xi_1}\psi_1(\varepsilon^\alpha X_1)S(X_1, y_2)\int_{\mathbb{R}^p} e^{i\Phi_1(y_1,\xi_1)}({}^tB_1(D_{y_1}))^N \big(y_i\psi_1(\xi_1 \\ &\quad -\nabla\phi_{z_1}(\varepsilon y_1))A_{z_1}(\varepsilon(y_1 - X_1))\big)\, dy_1\, d\xi_1\, dX_1 \\ &= \int\int_{\mathbb{R}^p\times\mathbb{R}^p} e^{i(x_1-y_1)\xi_1}\int_{\mathbb{R}^p} e^{i\frac{\phi_{z_1}(\varepsilon(y_1-X_1))-\phi_{z_1}(0)}{\varepsilon}} \\ &\quad \times\psi_1(\varepsilon^\alpha X_1)S(X_1, y_2)g_N\, dX_1\, dy_1\, d\xi_1, \end{aligned} \tag{3.27}$$

where

$$g_N = ({}^tB_1(D_{y_1}))^N \big(y_i\psi_1(\xi_1 - \nabla\phi_{z_1}(\varepsilon y_1))A_{z_1}(\varepsilon(y_1 - X_1))\big)$$

and ${}^tB_1(D_{y_1})$ is the transposed operator of $B_1(D_{y_1})$. Note that $g_N = O(\varepsilon^{N-1})$.

On the other hand, denote

$$\Phi_2(x_1, y_1, \xi_1) = (x_1 - y_1)\xi_1 + \frac{\phi_{z_1}(\varepsilon(y_1 - X_1)) - \phi_{z_1}(0)}{\varepsilon},$$

$$B_2(D_{\xi_1}, D_{y_1}) = \frac{(x_1 - y_1)\cdot D_{\xi_1} + (\nabla\phi_{z_1}(\varepsilon(y_1 - X_1)) - \xi_1)\cdot D_{y_1}}{|x_1 - y_1|^2 + |\nabla\phi_{z_1}(\varepsilon(y_1 - X_1)) - \xi_1|^2}.$$

Observing that

$$B_2(D_{\xi_1}, D_{y_1})e^{i\Phi_2(x_1,y_1,\xi_1)} = e^{i\Phi_2(x_1,y_1,\xi_1)},$$

again by (3.26) and integration by parts, we arrive at

$$k_1^\varepsilon = \int_{\mathbb{R}^p} \psi_1(\varepsilon^\alpha X_1) S(X_1, y_2) \int\int_{\mathbb{R}^p \times \mathbb{R}^p} e^{i\Phi_2(x_1, y_1, \xi_1)} \times ({}^t B_2(D_{\xi_1}, D_{y_1}))^M g_N \, dy_1 \, d\xi_1 \, dX_1. \tag{3.28}$$

By a trivial calculation, we find

$$|({}^t B_2(D_{\xi_1}, D_{y_1}))^M g_N| \leqslant \frac{\varepsilon^{N-1} |y_1| C_1(\xi_1 - \nabla\phi_{z_1}(\varepsilon y_1)) C_2(z_1 - \varepsilon y_1) C_3(z_1 + \varepsilon(y_1 - X_1))}{\left(1 + |x_1 - y_1|^2 + |\xi_1 - \nabla\phi_{z_1}(\varepsilon(y_1 - X_1))|^2\right)^{\frac{M}{2}}},$$

where $C_i(\cdot)$ are non-negative functions, and $C_1(\cdot)$, $C_3(\cdot)$ have compact support. Therefore by summing up the above, we have

$$|k_1^\varepsilon| \leqslant C\varepsilon^{N-1} \int\int_{\mathbb{R}^p \times \mathbb{R}^p} \frac{|S(X_1, y_2)| |y_1| C_3(z_1 + \varepsilon(y_1 - X_1))}{\left(1 + |x_1 - y_1|^2\right)^{\frac{M}{2}}} \, dX_1 \, dy_1. \tag{3.29}$$

Exactly as the proof of (3.29), we can derive a similar estimate for k_2^ε. By summing up (3.25), (3.29) and Minkowski inequality, we get

$$\begin{aligned} &\|(|x_1| + |y_2|) v_1^\varepsilon\| \\ &\leqslant C\varepsilon^{N-1} \times \left\| \left(\iint_{\mathbb{R}^p \times \mathbb{R}^{d-p}} \frac{\langle y_2\rangle^2 |S(X_1, y_2)|^2 |y_1|^2 C_3^2(z_1 + \varepsilon(y_1 - X_1))}{\left(1 + |x_1 - y_1|^2\right)^M} dx_1 \, dy_2 \right)^{\frac{1}{2}} \right\|_{L^1_{X_1, y_1}} \\ &\leqslant C\varepsilon^{N-1-p} \left\| \left(\int_{\mathbb{R}^{d-p}} \langle y_2\rangle^2 |S(X_1, y_2)|^2 |Z_1 - z_1 + \varepsilon X_1|^2 C_3^2(Z_1) \, dy_2 \right)^{\frac{1}{2}} \right\|_{L^1_{X_1, Z_1}} \\ &\leqslant C_{z_1} \varepsilon^{N-1-p}. \end{aligned} \tag{3.30}$$

On the other hand, we can rewrite v_2^ε in the following form:

$$\varepsilon^{\alpha N} \int\int_{\mathbb{R}^p \times \mathbb{R}^p} e^{i(x_1 - y_1)\xi_1} \psi_1(\xi_1 - \nabla\phi_{z_1}(\varepsilon y_1)) \int_{\mathbb{R}^p} e^{i\frac{\phi_{z_1}(\varepsilon(y_1 - X_1)) - \phi_{z_1}(\varepsilon y_1)}{\varepsilon}} \times \tilde{\psi}_2(\varepsilon^\alpha X_1) A_{z_1}(\varepsilon(y_1 - X_1)) X_1^N S(X_1, y_2) \, dX_1 \, dy_1 \, d\xi_1, \tag{3.31}$$

where $\tilde{\psi}_2(X_1) = X_1^{-N}\psi_2(X_1)$. Then exactly as the proof of (3.30), we can prove that

$$\|(|x_1| + |y_2|)v_2^\varepsilon\| \leqslant C_{z_1}\varepsilon^{\alpha N-1-p}. \tag{3.32}$$

Summing up (3.16), (3.24), (3.30) and (3.32) together, we find

$$\|w_{1,1}^\varepsilon\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} \leqslant C_{z_1}\varepsilon^{\alpha N-1-p} \tag{3.33}$$

for any positive number $\alpha < 1$.

Step 2: The estimate of $w_{1,2}^\varepsilon$.

Introduce another cut-off $\rho_1(y) \in C^\infty(\mathbb{R}^d)$ such that $0 \leqslant \rho_1 \leqslant 1$ and

$$\rho_1(y) = \begin{cases} 1, & \langle y_1\rangle \leqslant M\langle y_2\rangle, \\ 0, & \langle y_1\rangle \geqslant 2M\langle y_2\rangle. \end{cases}$$

Set $\rho_2(y) = 1 - \rho_1(y)$. Put

$$w_{1,2}^j = R_\varepsilon(E + i\kappa)\chi_1(P_\varepsilon)\Psi_2(z_1,\varepsilon)\left(\rho_j W_{z_1}^\varepsilon\right), \tag{3.34}$$

where $M > 1$ is to be taken large enough later. Then, by the uniform resolvent estimate in Besov space, and imbedding theorem from the weighted L^2 space to Besov space, for any $s > \frac{1}{2}$, we obtain

$$\begin{aligned} \|w_{1,2}^1\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} &\leqslant C\|\rho_1 W_{z_1}^\varepsilon\|_{B_{\frac{1}{2}}(\mathbb{R}^d)} \\ &\leqslant C\left(\int_{\mathbb{R}^d}\langle y\rangle^{2s}\rho_1(y)^2(W_{z_1}^\varepsilon(y_1,y_2))^2\,dy_1\,dy_2\right)^{\frac{1}{2}} \\ &\leqslant C\|A\|_{L^\infty}\left(\int_{\mathbb{R}^{d-p}}\langle y_2\rangle^{2s+p}\left(\int_{\mathbb{R}^p}|S(X_1,y_2)|\,dX_1\right)^2 dy_2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.35}$$

Finally, for the piece $w_{1,2}^2$, by functional calculus of pseudo-differential operators, we note that $\chi_1(P_\varepsilon)\Psi_2(z_1,\varepsilon)\rho_2$ is a pseudo-differential operator whose symbol is supported in $\operatorname{supp}\Big(\chi_1(\xi^2 + V_{z_1}(\varepsilon y))\psi_2(\xi_1 - \nabla\phi_{z_1}(\varepsilon y_1))\rho_2(y)\Big)$. In this set, one has

$$|\xi_1 - \nabla\phi_{z_1}(\varepsilon y_1)| \leqslant 2\delta, \quad |\xi^2 + V_{z_1}(\varepsilon y) - E| \leqslant 2\delta, \quad \frac{\langle y_1\rangle}{M\langle y_2\rangle} > 1.$$

By the assumption (A.4),

$$E - |\nabla \phi_{z_1}(\varepsilon y_1)|^2 - V_{z_1}(\varepsilon y) \geqslant \delta_0 > 0$$

for any ε, z_1 and y. It follows that for $0 < \delta << \delta_0$, $M >> 1$,

$$\xi_2^2 \geqslant \frac{\delta_0}{2}, \quad \sqrt{\frac{\delta_0}{2}} \leqslant |\xi| \leqslant C, \tag{3.36}$$

$$|y \cdot \xi| \leqslant |y_1|(|\xi_1| + |\xi_2|/M) \leqslant (1 - \eta_0)|y||\xi|, \quad \eta_0 > 0 \tag{3.37}$$

for (y, ξ) in the support of $\chi_1(\xi^2 + V_{z_1}(\varepsilon y))\psi_2(\xi_1 - \nabla\phi_{z_1}(\varepsilon y_1))\rho_2(y)$. We can thus construct a pseudo-differential operator $b_0(\varepsilon y, D)$ with symbol $b_0(y, \xi)$ supported in the region

$$\{|y| \leqslant R\} \cup \{(y, \xi); |y \cdot \xi| \leqslant (1 - \eta_0/2)|y||\xi|, |\xi| \geqslant c_0 > 0\} \tag{3.38}$$

and equal to 1 in a neighborhood of the support of

$$\chi_1(\xi^2 + V_{z_1}(\varepsilon y))\psi_2(\xi_1 - \nabla\phi_{z_1}(\varepsilon y_1))\rho_2(y),$$

such that the symbol of $(1 - b_0(\varepsilon y, D))\chi_1(P_\varepsilon)\Psi_2(z_1, \varepsilon)\rho_2$ is of the order

$$O(\langle y\rangle^{-N}) + O(\varepsilon^N \langle (z_1 + \varepsilon y_1, \varepsilon y_2)\rangle^{-N})$$

for $N > 1$ large enough. Since $\langle y_1\rangle^{-\tau} W_{z_1}^\varepsilon(y)$ is uniformly bounded in L^2 for $\tau > p/2$, applying (b) of Theorem 3.3, we obtain that for $s > \frac{p}{2} + 1$ and $s' > \frac{1}{2}$,

$$\begin{aligned} &\|w_{1,2}^2\|_{L^{2,-s}(\mathbb{R}^d)} \\ &\quad = \|R_\varepsilon(E + i\kappa)\,(b_0(\varepsilon y, D) + (1 - b_0(\varepsilon y, D)))\,\chi_1(P_\varepsilon)\Psi_2(z_1, \varepsilon)(\rho_2 W_{z_1}^\varepsilon)\|_{L^{2,-s}(\mathbb{R}^d)} \\ &\quad \leqslant C\{\|\langle y_1\rangle^{1-s} W_{z_1}^\varepsilon\| + \|\langle y_1\rangle^{-N+s'} W_{z_1}^\varepsilon\| + \varepsilon^N \|\langle y_1\rangle^{s'} \langle (z_1 + \varepsilon y_1)\rangle^{-N} W_{z_1}^\varepsilon\|\}, \end{aligned} \tag{3.39}$$

which together with (3.21) implies that

$$\|w_{1,2}^2\|_{L^{2,-s}(\mathbb{R}^d)} \leqslant C'(1 + \varepsilon^{N-p/2-s'}\langle z_1\rangle^N)\|A\|_{L^\infty}\|S\|_{L^2(R_{x_2}^{d-p}; L^1(R_{x_1}^d))}. \tag{3.40}$$

By summing up (3.22), (3.33), (3.35) and (3.40), we complete the proof of the Theorem. $\square$

4. The weak limit of $w^{\epsilon}_{z_1}$

Let $u_\varepsilon(x)$ be the unique solution to (1.1). For simplicity, we denote $u_\varepsilon(x+\frac{\varepsilon y}{2})$, $\overline{u_\varepsilon(x-\frac{\varepsilon y}{2})}$, $u_\varepsilon(x+\frac{\varepsilon y}{2})\overline{u_\varepsilon(x-\frac{\varepsilon y}{2})}$ by $\mathcal{U}_\varepsilon(x,y)$, $\tilde{\mathcal{U}}_\varepsilon(x,y)$ and $\mathcal{V}_\varepsilon(x,y)$, respectively. In 1932, Wigner [21] introduced the following transformation in quantum mechanics:

$$f_\varepsilon(x,\xi)=\frac{1}{(2\pi)^d}\int_{R^d} e^{-i\xi y}\mathcal{V}_\varepsilon(x,y)\,dy. \tag{4.1}$$

Now let us derive the equation satisfied by f_ε. Firstly, by (1.1), we have

$$i\,\frac{\alpha_\varepsilon}{\varepsilon}\mathcal{U}_\varepsilon+\triangle_x\mathcal{U}_\varepsilon+\frac{n^2(x+\frac{\varepsilon y}{2})}{\varepsilon^2}\mathcal{U}_\varepsilon=\mathcal{S}_\varepsilon\left(x+\frac{\varepsilon y}{2}\right).$$

Multiplying the above equation by $\tilde{\mathcal{U}}_\varepsilon$, we obtain

$$i\,\frac{\alpha_\varepsilon}{\varepsilon}\mathcal{V}_\varepsilon+\tilde{\mathcal{U}}_\varepsilon\triangle_x\mathcal{U}_\varepsilon+\frac{n^2(x+\frac{\varepsilon y}{2})}{\varepsilon^2}\mathcal{V}_\varepsilon=\mathcal{S}_\varepsilon\left(x+\frac{\varepsilon y}{2}\right)\tilde{\mathcal{U}}_\varepsilon. \tag{4.2}$$

A similar procedure yields the following equation:

$$-i\,\frac{\alpha_\varepsilon}{\varepsilon}\mathcal{V}_\varepsilon+\mathcal{U}_\varepsilon\triangle_x\tilde{\mathcal{U}}_\varepsilon+\frac{n^2(x-\frac{\varepsilon y}{2})}{\varepsilon^2}\mathcal{V}_\varepsilon=\overline{\mathcal{S}_\varepsilon(x-\frac{\varepsilon y}{2})}\mathcal{U}_\varepsilon. \tag{4.3}$$

Then by subtracting (4.3) from (4.2), we find

$$\begin{aligned}\alpha_\varepsilon\mathcal{V}_\varepsilon&+\frac{\varepsilon}{2i}\left(\tilde{\mathcal{U}}_\varepsilon\triangle_x\mathcal{U}_\varepsilon-\mathcal{U}_\varepsilon\triangle_x\tilde{\mathcal{U}}_\varepsilon\right)+\frac{1}{2i\varepsilon}\left(n^2\left(x+\frac{\varepsilon y}{2}\right)\right.\\&\left.-\,n^2\left(x-\frac{\varepsilon y}{2}\right)\right)\mathcal{V}_\varepsilon=\frac{\varepsilon}{2i}\left(\mathcal{S}_\varepsilon\left(x+\frac{\varepsilon y}{2}\right)\tilde{\mathcal{U}}_\varepsilon-\overline{\mathcal{S}_\varepsilon\left(x-\frac{\varepsilon y}{2}\right)}\mathcal{U}_\varepsilon\right).\end{aligned}$$

We take Fourier transform of the above equation with respect to y to get

$$\alpha_\varepsilon f_\varepsilon+\xi\cdot\nabla_x f_\varepsilon+\Theta^\varepsilon[n^2]f_\varepsilon=Q_\varepsilon, \tag{4.4}$$

where $\Theta^\varepsilon[n^2]f_\varepsilon$ is a pseudo-differential operator defined by

$$\begin{aligned}\Theta^\varepsilon[n^2]f_\varepsilon=&\frac{1}{2i\varepsilon(2\pi)^d}\int_{R^{2d}_{\eta,y}} e^{-iy(\xi-\eta)}\left(n^2\left(x+\frac{\varepsilon y}{2}\right)-n^2\left(x-\frac{\varepsilon y}{2}\right)\right)\\&\times f_\varepsilon(x,\eta)\,d\eta\,dy\end{aligned} \tag{4.5}$$

and

$$Q_\varepsilon = \frac{\varepsilon}{2i}\mathcal{F}_{y\to\xi}\left(\mathcal{S}_\varepsilon\left(x+\frac{\varepsilon y}{2}\right)\tilde{\mathcal{U}}_\varepsilon - \overline{\mathcal{S}_\varepsilon\left(x-\frac{\varepsilon y}{2}\right)}\mathcal{U}_\varepsilon\right). \tag{4.6}$$

Note by (4.1) that although $u_\varepsilon(x)$ is a complex-valued function, $f_\varepsilon(x,\xi)$ is a real-valued one, but it may change signs. However, when $\{u_\varepsilon\}$ is uniformly bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, there exists a subsequence of $\{f_\varepsilon\}$ which converges to a non-negative Radon measure, which is the so-called semi-classical measure, or Wigner measure. See [3,8,11]. We should point out that H-measure or micro-local defect measure has a very close relation with the semi-classical measure, one can, see [7,17] for more details.

If we replace the u_ε in (4.1) by the solutions to the evolutive Schrödinger equation, the limit equation to the correspondence of (4.4), the Wigner equation, was rigorously justified in [11] when the Schrödinger equation is linear. The nonlinear case was open since then. In [24], the authors proved that for 1-D Schrödinger–Poisson equation, the limit equation to the corresponding Wigner equation is actually 1-D Vlasov–Poisson equation, and the second author of the present work [23] proved a local result for the multi-dimensional case.

To present precisely the limit of Wigner transform of u_ε in our case, we first recall the test function space X_λ (see [2,5,11]): the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ under the norm:

$$\|\varphi\|_{X_\lambda} := \int_{\mathbb{R}^d}\sup_x\left\{\langle x,y\rangle^\lambda|(\mathcal{F}_{\xi\to y}\varphi)(x,y)|\right\}dy,$$

where we denote $\langle x,y\rangle = (1+|x|^2+|y|^2)^{\frac{1}{2}}$ and $(\mathcal{F}_{\xi\to y}\varphi)(x,y)$ the partial Fourier transform of $\varphi(x,\xi)$ with respect to ξ. The space X_λ is a Banach space with dual X_λ^*. Then an immediate consequence of Theorem 3.1 is

Theorem 4.1. *Under the assumptions* (A.1)–(A.3) *in* §3, *for any* $\lambda > 1$, *the family of Wigner transforms* f_ε *of* u_ε *is bounded in* X_λ^*, *and* $\{f_\varepsilon\}$ *admits a subsequence converging* $*$*-weakly to some non-negative*, *locally bounded Radon measure* f *such that*

$$\sup_{R>1}\frac{1}{R^\lambda}\int_{|x|\leqslant R}\int_{\xi\in\mathbb{R}^d} f(x,\xi)\,dx\,d\xi \leqslant C_\lambda\|A\|^2_{H^{1,s}(\mathbb{R}^p)}\|\langle x_1\rangle\langle x\rangle^s S\|^2_{L^1(\mathbb{R}^p_{x_1},H^1(\mathbb{R}^{d-p}))}$$

for any $s > \frac{1}{2}$.

Proof. The proof of this Theorem is the same as that of Theorem 3.1.1 in [2], but with a correction to a small error there. For any $\delta > 0$, we have

$$\|\langle x\rangle^{-\frac{1}{2}-\delta}u_\varepsilon\|_{L^2(\mathbb{R}^d)} \leqslant C_\delta\|u_\varepsilon\|_{B^*_{\frac{1}{2}}}. \tag{4.7}$$

For $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$, applying (4.7) and Hölder inequality, we get

$$\left|\int_{\mathbb{R}^{2d}} f_\varepsilon(x,\xi)\varphi(x,\xi)\,dx\,d\xi\right| \leqslant C_\delta^2 \|u_\varepsilon\|^2_{B^*_{\frac{1}{2}}} \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} \langle |x|+|y|\rangle^{1+2\delta} |\mathcal{F}_{\xi\to y}\varphi(x,y)|\,dy.$$

This bound and Theorem 3.1 imply that the family f_ε is bounded in the space $X^*_{1+2\delta}$ for any $\delta > 0$. From this and a similar argument in [8,11], we can extract a subsequence from $\{f_\varepsilon\}$ which converges $*$-weakly to a non-negative Radon measure f satisfying

$$\begin{aligned}\left|\int_{\mathbb{R}^{2d}} f(x,\xi)\varphi(x,\xi)\,dx\,d\xi\right| &\leqslant C_\delta^2 \|A\|^2_{H^{1,s}(\mathbb{R}^p)} \|\langle x_1\rangle\langle x\rangle^s S\|^2_{L^1(\mathbb{R}^p_{x_1}, H^1(\mathbb{R}^{d-p}))} \\ &\quad \times \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} \langle |x|+|y|\rangle^{1+2\delta} |\mathcal{F}_{\xi\to y}\varphi(x,y)|\,dy. \end{aligned} \tag{4.8}$$

Now setting $\varphi(x,\xi) = \frac{1}{R^{1+2\delta}}\chi(|x|\leqslant R)e^{-\mu|\xi|^2}$ in (4.8) and then taking the limit $\mu \to 0$, we obtain

$$\begin{aligned}&\frac{1}{R^{1+2\delta}} \int_{|x|\leqslant R}\int_{\mathbb{R}^d} f(x,\xi)\,dx\,d\xi \\ &\quad \leqslant C'_\delta \|A\|^2_{H^{1,s}(\mathbb{R}^p)} \|\langle x_1\rangle\langle x\rangle^s S\|^2_{L^1(\mathbb{R}^p_{x_1}, H^1(\mathbb{R}^{d-p}))}, \quad R > 1,\end{aligned}$$

from which we conclude the proof of this Theorem. $\square$

Note that the decay of $n(x)^2 - E$ at the infinity is not needed in Theorem 4.1. In particular, it holds for generalized N-body-type refraction index. As is well-known from [11], the formal limit of $\Theta^\varepsilon[n^2]f_\varepsilon$ equals $\frac{1}{2}\nabla n^2 \cdot \nabla_\xi f$. To determine the limit of source $Q_\varepsilon(x,\xi)$ in (4.4) as $\varepsilon \to 0$, similar to [2,5], it is useful to introduce the re-scaled function $w^\varepsilon_{z_1}(y)$ (see (3.12)). As commented by the authors in [5], this function measures the concentration of u_ε on R^p close to the point $(z_1, 0)$, and it also carries the relevant oscillations of u_ε at this point.

Note that

$$-i\varepsilon\alpha_\varepsilon w^\varepsilon_{z_1}(y) - \triangle w^\varepsilon_{z_1}(y) - n^2_{z_1}(\varepsilon y) w^\varepsilon_{z_1}(y) = W^\varepsilon_{z_1}(y), \tag{4.9}$$

where $n_{z_1}(y) = n(z_1 + y_1, y_2)$. One expects formally that $w^\varepsilon_{z_1}(y)$ converges toward the solution $w_{z_1}(y)$ to

$$-0i w_{z_1}(y) - \triangle w_{z_1}(y) - n^2_{z_1}(0) w_{z_1}(y) = W_{z_1}(y), \tag{4.10}$$

where

$$W_{z_1}(y) = \lim_{\varepsilon\to 0} W^{\varepsilon}_{z_1}(y) = A_{z_1}(0)\int_{R^p} e^{i\nabla\phi_{z_1}(0)(y_1-X_1)} S(X_1, y_2)\, dX_1$$

and $-0i w_{z_1}(y)$ means that $w_{z_1}(y)$ should satisfy the radiation condition at infinity (see [2,5]). As pointed by the authors in [5], the deep difficulty in passing the limit from (4.9) to (4.10) is one of the reasons why they restrict themselves to the constant refraction index there.

To give a rigorous proof to the limit from (4.9) to (4.10), we introduce the following intermediate equation:

$$-i\varepsilon\alpha_\varepsilon \overline{w}^{\varepsilon}_{z_1} - \triangle \overline{w}^{\varepsilon}_{z_1}(y) - n^2_{z_1}(\varepsilon y)\overline{w}^{\varepsilon}_{z_1}(y) = W_{z_1}(y). \tag{4.11}$$

Now let us first compare the difference between $w^{\varepsilon}_{z_1}$ with $\overline{w}^{\varepsilon}_{z_1}$.

Lemma 4.2. *Let* $w^{\varepsilon}_{z_1}, \overline{w}^{\varepsilon}_{z_1}$ *be solutions to* (4.9) *and* (4.11), *respectively. Denote* $q^{\varepsilon}_1(y) = w^{\varepsilon}_{z_1} - \overline{w}^{\varepsilon}_{z_1}$. *Then under the assumptions* (1.3), (1.4), (A.4) *and* (A.5), *for any* $s > \frac{p}{2}+2$, *there holds*

$$\|q^{\varepsilon}_1\|_{L^{2,-s}(\mathbb{R}^d)} \leqslant C\varepsilon, \tag{4.12}$$

where the constant C *only depends on* z_1, A *and* ϕ, *but is independent of* ε.

Proof. From (4.9) and (4.11), we obtain

$$-i\varepsilon\alpha_\varepsilon q^{\varepsilon}_1 - \Delta q^{\varepsilon}_1 - n^2_{z_1}(\varepsilon y) q^{\varepsilon}_1 = V^{\varepsilon}_1 + V^{\varepsilon}_2, \tag{4.13}$$

where

$$V^{\varepsilon}_1(y) = \int_{\mathbb{R}^p} e^{i\frac{\phi_{z_1}(\varepsilon(y_1-X_1))-\phi_{z_1}(0)}{\varepsilon}} (A_{z_1}(\varepsilon(y_1-X_1)) - A_{z_1}(0)) S(X_1, y_2)\, dX_1,$$

$$V^{\varepsilon}_2(y) = A_{z_1}(0)\int_{\mathbb{R}^p} \left(e^{i\frac{\phi_{z_1}(\varepsilon(y_1-X_1))-\phi_{z_1}(0)}{\varepsilon}} - e^{i\nabla\phi_{z_1}(0)\cdot(y_1-X_1)}\right) S(X_1, y_2)\, dX_1.$$

It is trivial to note that

$$|V^{\varepsilon}_1(y)| \leqslant \varepsilon \|\nabla A\|_{L^\infty} S_1(y), \tag{4.14}$$

$$|V^{\varepsilon}_2(y)| \leqslant \varepsilon \|A\|_{L^\infty} \|\nabla^2\phi\|_{L^\infty} S_2(y), \tag{4.15}$$

where

$$S_j(y) = \int_{\mathbb{R}^p} (|y_1| + |X_1|)^j |S(X_1, y_2)| \, dX_1$$

therefore for any $s > \frac{p}{2} + 2$, there holds

$$\begin{aligned} \|V_i^\varepsilon\|_{L^{2,-s}(\mathbb{R}^d)} &\leqslant C\varepsilon \|A\|_{C^1} \langle \|\nabla^2 \phi\|_{L^\infty} \rangle \|\langle y \rangle^{-s} S_2(y)\|_{L^2} \\ &\leqslant C\varepsilon \|A\|_{C^1} \langle \|\nabla^2 \phi\|_{L^\infty} \rangle \|\langle y_1 \rangle^2 S(y)\|_{L^2(\mathbb{R}^{d-p}_{y_2}; L^1(\mathbb{R}^p_{y_1}))}. \end{aligned} \tag{4.16}$$

With the above information, we can follow step by step the proof of Theorem 3.4. Using the same notation as that in the proof of Theorem 3.4, we now outline the proof of (4.12). Firstly by the proof of (3.22), we obtain

$$\|\chi_2(P_\varepsilon) q_1^\varepsilon\|_{L^{2,-s}(\mathbb{R}^d)} \leqslant C\varepsilon \|A\|_{C^1} \langle \|\nabla^2 \phi\|_{L^\infty} \rangle \|\langle y_1 \rangle^2 S(y)\|_{L^2(\mathbb{R}^{d-p}_{y_2}; L^1(\mathbb{R}^p_{y_1}))}. \tag{4.17}$$

Corresponding to the decomposition in (3.23), we decompose $\chi_1(P_\varepsilon) q_1^\varepsilon$ by the following:

$$R_\varepsilon(E + i\kappa) \chi_1(P_\varepsilon) \left(\Psi_1(z_1, \varepsilon) + \Psi_2(z_1, \varepsilon) \right) q_1^\varepsilon \triangleq q_{1,1}^\varepsilon + q_{1,2}^\varepsilon.$$

Exactly similar to the proof of (3.33), we can prove that

$$\|q_{1,1}^\varepsilon\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} \leqslant C_{z_1} \varepsilon^{\alpha N - 1 - p} \tag{4.18}$$

for any positive number $\alpha < 1$.

Similarly to the decomposition in (3.34), we decompose $q_{1,2}^\varepsilon$ as

$$q_{1,2}^j = R_\varepsilon(E + i\kappa) \chi_1(P_\varepsilon) \Psi_2(z_1, \varepsilon) \left(\rho_j (V_1^\varepsilon + V_2^\varepsilon) \right). \tag{4.19}$$

Then by an argument used in the proof of (3.35) and (4.14), (4.15), we arrive at

$$\|q_{1,2}^1\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} \leqslant C\varepsilon \|A\|_{C^1} \langle \|\nabla^2 \phi\|_{L^\infty} \rangle \|\langle y_2 \rangle^{\frac{s+p}{2}} \langle y_1 \rangle^2 S(y)\|_{L^2(\mathbb{R}^{d-p}_{y_2}; L^1(\mathbb{R}^p_{y_1}))} \tag{4.20}$$

for any $s > 5$. Observe that the proof of (3.40) also yields

$$\|q_{1,2}^2\|_{L^{2,-s}(\mathbb{R}^d)} \leqslant C\varepsilon \left(1 + \varepsilon^{N - \frac{p}{2} - s'} \langle z_1 \rangle^N \right), \tag{4.21}$$

which completes the proof the lemma. □

Next let us turn to comparing the solution of (4.11) with that of (4.10), which is the key step in the limit from (4.9) to (4.10).

Lemma 4.3. *Let* $w_{z_1}, \overline{w}^{\varepsilon}_{z_1}$ *be solutions to* (4.10) *and* (4.11), *respectively, and* $q_2^{\varepsilon}(y) = \overline{w}^{\varepsilon}_{z_1} - w_{z_1}$. *Assume that there hold* (1.3), (1.4), (A.5) *and*

(A.6)

$$E - |\nabla \phi_{z_1}(0)|^2 - V_{z_1}(\varepsilon y) \geqslant \delta_0. \tag{4.22}$$

Then for any $s > \frac{p}{2} + 2$, *there exists a subsequence of* $\{q_2^{\varepsilon}\}$, *which we still denote by* $\{q_2^{\varepsilon}\}$, *such that*

$$q_2^{\varepsilon} \rightharpoonup 0 \quad in\ L^{2,-s}(\mathbb{R}^d) \tag{4.23}$$

as $\varepsilon \to 0$.

Remark 4.4. Note that the δ_0 in (4.22) can be a positive constant which depends on z_1 locally uniformly in $\mathbb{R}^p$. In order to use the same notation as that in the proof of Theorem 3.4, we still denote it by δ_0 here.

Proof. Let

$$R_{\varepsilon}(i\kappa) = (-\Delta - n^2_{z_1}(\varepsilon y) - i\kappa)^{-1}, \quad R_0(i\kappa) = (-\Delta - n^2_{z_1}(0) - i\kappa)^{-1} \tag{4.24}$$

with $\kappa = \varepsilon\alpha_{\varepsilon}$. Set $\tilde{n}_{z_1,\varepsilon} = n^2_{z_1}(\varepsilon y) - n^2_{z_1}(0)$. Then we can rewrite q_2^{ε} as

$$q_2^{\varepsilon} = \left(R_{\varepsilon}(i\kappa)\tilde{n}_{z_1,\varepsilon}R_0(i\kappa)\right) W_{z_1}. \tag{4.25}$$

Let $\delta > 0$, $\zeta(\xi) \in C^{\infty}(\mathbb{R}^d)$ such that $\zeta(\xi) = 1$ on $\{\xi : |\, |\xi| - |n_{z_1}(0)|\, | > \delta\}$, and with $\operatorname{supp}\zeta(\cdot) \subset \{\xi : |\, |\xi| - |n_{z_1}(0)|\, | > \frac{\delta}{2}\}$. Then we can decompose q_2^{ε} as

$$\begin{aligned} q_2^{\varepsilon} &= \left\{R_{\varepsilon}(i\kappa)\tilde{n}_{z_1,\varepsilon}R_0(i\kappa)(\zeta(D) + (1 - \zeta(D)))\right\} W_{z_1} \\ &\triangleq q^{\varepsilon}_{2,1} + q^{\varepsilon}_{2,2}. \end{aligned} \tag{4.26}$$

In the subsequence, for a clearer presentation, we divide the proof into three main steps.

Step 1: We want to prove that

$$\|q^{\varepsilon}_{2,1}\|_{L^{2,-s}(\mathbb{R}^d)} \to 0 \quad \text{for} \quad s > \frac{p}{2} + 1. \tag{4.27}$$

Step 1.1: For simplicity, we denote $\mathcal{H}_\varepsilon^1 = \tilde{n}_{z_1,\varepsilon} R_0(i\kappa)\zeta(D)W_{z_1}$. Then with the same notation as that in the proof of Theorem 3.4, we decompose $q_{2,1}^\varepsilon$ as

$$q_{2,1}^\varepsilon = R_\varepsilon(i\kappa)(\chi_1(P_\varepsilon) + \chi_2(P_\varepsilon))\mathcal{H}_\varepsilon^1 \triangleq \mathcal{A}_1^\varepsilon + \mathcal{A}_2^\varepsilon. \tag{4.28}$$

Note that $R_0(i\kappa)\zeta(D)W_{z_1}$ is uniformly bounded in $B^*_{\frac{p}{2}}$, and that for any $\delta > 0$,

$$\|\tilde{n}_{z_1,\varepsilon}\langle y\rangle^{-\delta}\|_{L^\infty(\mathbb{R}^d)} \to 0$$

as $\varepsilon \to 0$, so that $\tilde{n}_{z_1,\varepsilon}\langle y\rangle^{-\delta} \to 0$ as operator on $L^2(\mathbb{R}^d)$. Therefore for any $s > \frac{p}{2}$, the argument used in (3.22) yields

$$\begin{aligned}\|\mathcal{A}_2^\varepsilon\|_{B_s^*(\mathbb{R}^d)} &\leqslant C\|\mathcal{H}_\varepsilon^1\|_{B_s^*(\mathbb{R}^d)} \\ &\leqslant \|\tilde{n}_{z_1,\varepsilon}\langle y\rangle^{\frac{p}{2}-s}\|_{L^\infty}\|R_0(i\kappa)\zeta(D)W_{z_1}\|_{B^*_{\frac{p}{2}}(\mathbb{R}^d)} \to 0\end{aligned} \tag{4.29}$$

as $\varepsilon \to 0$. To prove that $\mathcal{A}_1^\varepsilon \to 0$ as $\varepsilon \to 0$, we need to repeat the argument used in the proof of Theorem 3.4. For completeness, we present the details here.

Step 1.2: Firstly corresponding to the decomposition in (3.23), we decompose $\mathcal{A}_1^\varepsilon$ further as

$$\mathcal{A}_1^\varepsilon = R_\varepsilon(i\kappa)\chi_1(P_\varepsilon)(\Psi_1(z_1,0) + \Psi_2(z_1,0))\mathcal{H}_\varepsilon^1 \triangleq \mathcal{A}_{1,1}^\varepsilon + \mathcal{A}_{1,2}^\varepsilon, \tag{4.30}$$

where $\Psi_j(z_1,0)$, $j = 1,2$, is the pseudo-differential operator defined below (3.23) but with $\varepsilon = 0$.

To use the stationary phase method to estimate (4.30), we define the phase functions Φ_3 and Φ_4 as

$$\Phi_3 = (\eta_1 - \xi_1)y_1 + (\nabla\phi_{z_1}(0) - \eta_1)Y_1, \quad \Phi_4 = x_1\xi_1 + (y_2 - Y_2)\eta_2 - \nabla\phi_{z_1}(0)X_1.$$

Notice that pseudo-differential operator is defined via oscillatory integral, we can rewrite $\Psi_1(z_1,0)\mathcal{H}_\varepsilon^1$ as

$$\begin{aligned}\Psi_1(z_1,0)\mathcal{H}_\varepsilon^1 = -A_{z_1}(0)\lim_{\nu\to 0}\int_{\mathbb{R}^{2d}} e^{i\Phi_4}\psi_1(\xi_1 - \nabla\phi_{z_1}(0))S(X_1,Y_2) \\ \times\left(\int_{\mathbb{R}^{3p}} e^{i\Phi_3}\frac{\zeta(\eta)\tilde{n}_{z_1,\varepsilon}e^{-\frac{\nu|Y_1|^2}{2}}}{i\varepsilon\alpha_\varepsilon - |\eta|^2 + n_{z_1}^2(0)}\,dy_1\,dY_1\,d\eta_1\right)d\xi_1\,d\eta_2\,dX_1\,dY_2,\end{aligned}$$

which together with the fact

$$\int_{\mathbb{R}^p} e^{i(\nabla\phi_{z_1}(0)-\eta_1)Y_1} e^{-\frac{v|Y_1|^2}{2}}\,dY_1 = v^{-\frac{n}{2}} e^{-\frac{|\nabla\phi_{z_1}(0)-\eta_1|^2}{2v}},$$

leads to

$$\begin{aligned}\Psi_1(z_1,0)\mathcal{H}_\varepsilon^1 = -A_{z_1}(0)&\int_{\mathbb{R}^{2d}} e^{i\Phi_4}\psi_1(\xi_1-\nabla\phi_{z_1}(0))\\ &\times\frac{\zeta(\nabla\phi_{z_1}(0),\eta_2)S(X_1,Y_2)}{i\varepsilon\alpha_\varepsilon-|\nabla\phi_{z_1}(0)|^2-|\eta_2|^2+n_{z_1}^2(0)}\\ &\times\left(\int_{\mathbb{R}^p} e^{i(\nabla\phi_{z_1}(0)-\xi_1)y_1}\tilde{n}_{z_1,\varepsilon}\,dy_1\right)d\xi_1\,d\eta_2\,dX_1\,dY_2. \end{aligned} \tag{4.31}$$

Since on the support of $\psi_1(\xi_1-\nabla\phi_{z_1}(0))$, $|\xi_1-\nabla\phi_{z_1}(0)|\geqslant\delta$, we define the differential operator $B_3(D_{y_1})$ as

$$B_3(D_{y_1}) = \frac{(\nabla\phi_{z_1}(0)-\xi_1)\cdot D_{y_1}}{|\nabla\phi_{z_1}(0)-\xi_1|^2}.$$

Note that

$$B_3 e^{i(\nabla\phi_{z_1}(0)-\xi_1)y_1} = e^{i(\nabla\phi_{z_1}(0)-\xi_1)y_1},$$

we use integration by parts in the term $\int_{\mathbb{R}^p} e^{i(\nabla\phi_{z_1}(0)-\xi_1)y_1}\tilde{n}_{z_1,\varepsilon}\,dy_1$ of (4.31) to get

$$\Psi_1(z_1,0)\mathcal{H}_\varepsilon^1 = \left(\Psi_1(z_1,0)\left(({}^tB_3)^N\tilde{n}_{z_1,\varepsilon}\right)R_0(i\kappa)\zeta(D)\right)W_{z_1}, \tag{4.32}$$

where ${}^tB_3$ denotes the transposition of B_3. But by (1.4), we have

$$|({}^tB_3)^N\tilde{n}_{z_1,\varepsilon}|\leqslant\frac{1}{\delta^N}\,|\nabla_{y_1}^N\tilde{n}_{z_1,\varepsilon}|\leqslant C_\delta\varepsilon^N\langle z_1+\varepsilon y\rangle^{-(N+\rho_0)}.$$

Hence,

$$\begin{aligned}&\|({}^tB_3)^N(\tilde{n}_{z_1,\varepsilon})(R_0(i\kappa)\zeta(D))W_{z_1}\|_{B_{\frac{1}{2}}(\mathbb{R}^d)}\\ &\quad\leqslant\|\langle y\rangle^{\frac{p}{2}+\frac{1}{2}+0}({}^tB_3)^N\tilde{n}_{z_1,\varepsilon}\langle y\rangle^{-\frac{p}{2}-0}R_0(i\kappa)\zeta(D)W_{z_1}\|\\ &\quad\leqslant C_\delta\varepsilon^N\|\langle y\rangle^{\frac{p}{2}+\frac{1}{2}+0}\langle z_1+\varepsilon y_1\rangle^{-(N+\rho_0)}\|_{L^\infty(\mathbb{R}^d)}\leqslant C\varepsilon^{N-\frac{p}{2}-\frac{1}{2}-0}, \end{aligned} \tag{4.33}$$

uniformly for z_1 in any compact subset of $\mathbb{R}^p$. Then by (a) of Theorem 3.3, we arrive at

$$\|\mathcal{A}^{\varepsilon}_{1,1}\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} \leqslant C\|\Psi_1(z_1,0)\mathcal{H}^1_{\varepsilon}\|_{B_{\frac{1}{2}}(\mathbb{R}^d)} \leqslant C\varepsilon^{N-\frac{p}{2}-\frac{1}{2}-0}, \tag{4.34}$$

which tends to 0 when we take $N > \frac{p}{2} + \frac{1}{2}$.

Step 1.3: Corresponding to the decomposition in (3.34), we set

$$\mathcal{A}^j_{1,2} = R_{\varepsilon}(i\kappa)\chi_1(P_{\varepsilon})\Psi_2(z_1,0)(\rho_j\mathcal{H}^1_{\varepsilon}), \quad j=1,2, \tag{4.35}$$

where ρ_j is the same cut-off function as that in (3.34). Since ζ is supported away from $|\xi|^2 - n^2_{z_1}(0) = 0$, $\langle x_2\rangle^M R_0(i\kappa)\zeta(D)\langle x_2\rangle^{-M}$ is uniformly bounded on Besov spaces for any $M > 0$ and therefore

$$\|\langle x_2\rangle^M R_0(i\kappa)\zeta(D)W_{z_1}\|_{B^*_{\frac{p}{2}}(\mathbb{R}^d)} \leqslant \|\langle y_2\rangle^M \int_{\mathbb{R}^p} |S(X_1,y_2)|\,dX_1\|. \tag{4.36}$$

Then by the uniform resolvent estimate in Besov space, for any $M > \frac{p}{2} + \frac{1}{2}$, we obtain

$$\begin{aligned}\|\mathcal{A}^1_{1,2}\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} &\leqslant C\|\rho_1\mathcal{H}^1_{\varepsilon}\|_{B_{\frac{1}{2}}(\mathbb{R}^d)}\\ &\leqslant C\|\langle y\rangle^{\frac{p}{2}+\frac{1}{2}+0-M}\rho_1(y)\tilde{n}_{z_1,\varepsilon}\langle y\rangle^{-\frac{p}{2}-0}\langle y_2\rangle^M R_0(i\kappa)\zeta(D)W_{z_1}\|\\ &\leqslant C\|\langle y\rangle^{\frac{p}{2}+\frac{1}{2}+0-M}\tilde{n}_{z_1,\varepsilon}\|_{L^{\infty}(\mathbb{R}^d)} \to 0\end{aligned} \tag{4.37}$$

as $\varepsilon \to 0$.

Finally with (A.6), similar to the argument used in the proof of (3.39), for any $s > \frac{p}{2} + 1$, we obtain

$$\begin{aligned}&\|\mathcal{A}^2_{1,2}\|_{L^{2,-s}(\mathbb{R}^d)}\\ &\quad = \|R_{\varepsilon}(i\kappa)(b_0(\varepsilon y,D) + (1-b_0(\varepsilon y,D)))\chi_1(P_{\varepsilon})\Psi_2(z_1,0)(\rho_2\mathcal{H}^1_{\varepsilon})\|_{L^{2,-s}(\mathbb{R}^d)}\\ &\quad \leqslant C\left\{\|\langle y\rangle^{1-s}\mathcal{H}^1_{\varepsilon}\| + \|\langle y\rangle^{-N}\mathcal{H}^1_{\varepsilon}\| + \varepsilon^N\|\langle y\rangle^{\frac{1}{2}+0}\langle(z_1+\varepsilon y_1,\varepsilon y_2)\rangle^{-N}\mathcal{H}^1_{\varepsilon}\|\right\},\end{aligned} \tag{4.38}$$

where $b_0(\varepsilon y, D)$ is the same pseudo-differential operator as that in (3.39). However as $s > \frac{p}{2} + 1$, there holds

$$\|\langle y\rangle^{1-s}\mathcal{H}^1_{\varepsilon}\| \leqslant C\|\langle y\rangle^{1-s+\frac{p}{2}+0}\tilde{n}_{z_1,\varepsilon}\|_{L^{\infty}}\|R_0(i\kappa)\zeta(D)W_{z_1}\|_{B^*_{\frac{p}{2}}(\mathbb{R}^d)},$$

which approaches 0 as $\varepsilon \to 0$. Similar argument can be used to prove that the other two terms in (4.38) tend to 0 as $\varepsilon \to 0$. Therefore

$$\|\mathcal{A}_{1,2}^2\|_{L^{2,-s}(\mathbb{R}^d)} \to 0 \quad \text{for} \quad s > \frac{p}{2} + 1. \tag{4.39}$$

Summing up Step 1.1 through Step 1.3, we conclude (4.27).

Step 2: Estimate of $q_{2,2}^{\varepsilon}$.

Firstly similar to the decomposition in (4.30), we split $q_{2,2}^{\varepsilon}$ further as

$$q_{2,2}^{\varepsilon} = R_{\varepsilon}(i\kappa)\tilde{n}_{z_1,\varepsilon}R_0(i\kappa)(1-\zeta(D))(\Psi_1(z_1,0)+\Psi_2(z_1,0))W_{z_1}. \tag{4.40}$$

Note that formally

$$\Psi_1(z_1,0)W_{z_1} = A_{z_1}(0)\int_{\mathbb{R}^{2p}} e^{i(x_1\xi_1 - \nabla\phi_{z_1}(0)X_1)}\psi_1(\xi_1 - \nabla\phi_{z_1}(0))S(X_1,y_2) \times\left(\int_{\mathbb{R}^p} e^{-i(\xi_1-\nabla\phi_{z_1}(0))y_1}\,dy_1\right)dX_1\,d\xi_1$$

and the Fourier transform of 1 equals $\delta(0)$, we obtain

$$\Psi_1(z_1,0)W_{z_1} = 0. \tag{4.41}$$

One can justify (4.41) rigorously via the argument used in the proof of (4.31).

Combining (4.40) with (4.41), we obtain

$$\begin{aligned} q_{2,2}^{\varepsilon} &= R_{\varepsilon}(i\kappa)\tilde{n}_{z_1,\varepsilon}R_0(i\kappa)(1-\zeta(D))\Psi_2(z_1,0)(\rho_1+\rho_2)W_{z_1} \\ &\triangleq \mathcal{B}_1^{\varepsilon} + \mathcal{B}_2^{\varepsilon}, \end{aligned} \tag{4.42}$$

where ρ_j is again the same cut-off function as that in (3.34).

Notice that by the resolvent equation, we can also rewrite $\mathcal{B}_1^{\varepsilon}$ as

$$\mathcal{B}_1^{\varepsilon} = (R_{\varepsilon}(i\kappa) - R_0(i\kappa))(1-\zeta(D))\Psi_2(z_1,0)(\rho_1 W_{z_1}).$$

Hence the proof of (3.34) implies that

$$\|\mathcal{B}_1^{\varepsilon}\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} \leqslant C\|A\|_{L^\infty}\left(\int_{\mathbb{R}^{d-p}}\langle y_2\rangle^{p+1+0}\left(\int_{\mathbb{R}^p}|S(X_1,y_2)|\,dX_1\right)^2 dy_2\right)^{\frac{1}{2}}. \tag{4.43}$$

Note that on the support of $(1-\zeta(\xi))\psi_2(\xi_1 - \nabla\phi_{z_1}(0))\rho_2(y)$, there holds (3.36) and (3.37). Then we can construct a pseudo-differential operator $b_0(\varepsilon y, D)$ with the properties listed below (3.37), and the symbol of $(1-\zeta(D))(1-b_0(\varepsilon y, D))\Psi_2(z_1,0)\rho_2(y)$

is of the order $O(\langle y\rangle^{-N}+\varepsilon^N\langle \varepsilon y\rangle^{-N})$ for N large enough. With the operator $b_0(\varepsilon y, D)$, we decompose $\mathcal{B}_2^\varepsilon$ further as

$$\begin{aligned}\mathcal{B}_2^\varepsilon &= R_\varepsilon(i\kappa)\tilde{n}_{z_1,\varepsilon}R_0(i\kappa)(1-\zeta(D))(b_0(\varepsilon y, D)+(1-b_0(\varepsilon y, D)))\\ &\quad\times\Psi_2(z_1,0)(\rho_2 W_{z_1})\triangleq\mathcal{B}_{2,1}^\varepsilon+\mathcal{B}_{2,2}^\varepsilon.\end{aligned}\tag{4.44}$$

Then

$$\begin{aligned}\|\mathcal{B}_{2,2}^\varepsilon\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} &= \|\,(R_\varepsilon(i\kappa)-R_0(i\kappa))\,(1-\zeta(D))(1-b_0(\varepsilon y, D))\\ &\qquad\times\Psi_2(z_1,0)(\rho_2 W_{z_1})\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)}\\ &\leqslant C\|(1-\zeta(D))(1-b_0(\varepsilon y, D))\Psi_2(z_1,0)(\rho_2 W_{z_1})\|_{B_{\frac{1}{2}}(\mathbb{R}^d)}\\ &\leqslant C\|(\langle y\rangle^{-N}+\varepsilon^N\langle\varepsilon y\rangle^{-N})\langle y\rangle^{\frac{1}{2}+0}W_{z_1}\|\leqslant C_{z_1}.\end{aligned}\tag{4.45}$$

To estimate $\mathcal{B}_{2,1}^\varepsilon$, we introduce a microlocal partition of unity in the form: $1=b_+(x,\xi)+b_-(x,\xi)$ for $(x,\xi)\in\mathbb{R}^{2n}$ with ξ in a neighborhood of $\mathrm{supp}(1-\zeta)$ such that

$$\begin{aligned}\mathrm{supp}\, b_+ &\subset \{y;\,|y|\leqslant R\}\cup\left\{(y,\xi),\, y\cdot\xi>-\left(1-\frac{\eta_0}{8}\right)|y||\xi|\right\},\\ \mathrm{supp}\, b_- &\subset \{y;\,|y|\leqslant R\}\cup\left\{(y,\xi),\;\; y\cdot\xi<-\left(1-\frac{\eta_0}{4}\right)|y||\xi|\right\},\end{aligned}\tag{4.46}$$

where η_0 is the same as that in (3.37). Then we decompose $\mathcal{B}_{2,1}^\varepsilon$ further as

$$\mathcal{B}_{2,1}^\varepsilon=\mathcal{B}_{2,1}^+ +\mathcal{B}_{2,1}^-\qquad\text{with}\tag{4.47}$$

$$\mathcal{B}_{2,1}^\pm=R_\varepsilon(i\kappa)\tilde{n}_{z_1,\varepsilon}b_\pm(\varepsilon y, D)(1-\zeta(D))R_0(i\kappa)b_0(\varepsilon y, D)\Psi_2(z_1,0)(\rho_2 W_{z_1})$$

$b_-(\varepsilon y, D)$ and $b_0(\varepsilon y, D)$ satisfy the conditions in (c) of Theorem 3.3. Therefore, by (a) and (c) of Theorem 3.3, we obtain

$$\begin{aligned}\|\mathcal{B}_{2,1}^-\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)} &\leqslant C\|\tilde{n}_{z_1,\varepsilon}\langle y\rangle^{-\delta}\|_{L^\infty(\mathbb{R}^d)}\|\langle y\rangle^{\frac{1}{2}+\delta}b_-(\varepsilon y, D)R_0(i\kappa)b_0(\varepsilon y, D)\langle y\rangle^{\frac{p}{2}+\delta}\|\\ &\quad\times\|\langle y\rangle^{-\frac{p}{2}-\delta}W_{z_1}\|\to 0\end{aligned}\tag{4.48}$$

as $\varepsilon\to 0$. Since the commutator $[\tilde{n}_{z_1,\varepsilon}, b_+(\varepsilon y, D)]$ is a pseudo-differential operator of the order $O(\varepsilon)$ with symbol supported in $\mathrm{supp}\, b_+$, for any $s>\frac{p}{2}+\delta$, we apply (b) of

Theorem 3.3 to get

$$\begin{aligned}\|\mathcal{B}^{+}_{2,1}\|_{L^{2,-(s+2)}(\mathbb{R}^d)} &\leqslant \Big(\|\langle y\rangle^{-s-2}R_\varepsilon(i\kappa)b_+(\varepsilon y,D)\langle y\rangle^{s+1}\|\|\langle y\rangle^{-\delta}\tilde{n}_{z_1,\varepsilon}\|_{L^\infty(\mathbb{R}^d)}\\ &\quad+\|\langle y\rangle^{-s-2}R_\varepsilon(i\kappa)[b_+(\varepsilon y,D),\tilde{n}_{z_1,\varepsilon}]\langle y\rangle^{s+1}\|\Big)\cdot\|\langle y\rangle^{-s-1+\delta}\\ &\quad\times(1-\zeta(D))R_0(i\kappa)b_0(\varepsilon y,D)\Psi_2(z_1,0)\langle y\rangle^{s-\delta}\|\|\langle y\rangle^{-s+\delta}W_{z_1}\|\\ &\leqslant C(\|\langle y\rangle^{-\delta}\tilde{n}_{z_1,\varepsilon}\|_{L^\infty(\mathbb{R}^d)}+\varepsilon)\|\langle y\rangle^{-s+\delta}W_{z_1}\|\to 0 \qquad (4.49)\end{aligned}$$

as $\varepsilon\to 0$.

By summing up (4.27), (4.42) to (4.49), we obtain the following decomposition for q_2^ε:

$$q_2^\varepsilon = r_1^\varepsilon + r_2^\varepsilon \qquad (4.50)$$

with

$$\|r_1^\varepsilon\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)}\leqslant C_{z_1},\quad \|r_2^\varepsilon\|_{L^{2,-s}(\mathbb{R}^d)}\to 0\quad\text{as}\quad\varepsilon\to 0$$

for $s>\frac{p}{2}+2$. Eq. (4.50) implies that $\{q_2^\varepsilon\}$ is uniformly bounded in $L^{2,-s}(\mathbb{R}^d)$ for any $s>\frac{p}{2}+2$. Therefore, there exists $q_2\in L^{2,-s}(\mathbb{R}^d)$ and a subsequence of $\{q_2^\varepsilon\}$, which we still denote by $\{q_2^\varepsilon\}$, such that

$$q_2^\varepsilon \rightharpoonup q_2\quad\text{weakly in}\quad L^{2,-s}(\mathbb{R}^d)\quad\text{as}\quad\varepsilon\to 0. \qquad (4.51)$$

Furthermore, (4.50) shows that for any $\psi'\in C_0^\infty(\mathbb{R}^d)$,

$$|(q_2,\psi')|\leqslant \sup_\varepsilon\|r_1^\varepsilon\|_{B^*_{\frac{1}{2}}}\|\psi'\|_{B_{\frac{1}{2}}}\leqslant C_{z_1}\|\psi'\|_{B_{\frac{1}{2}}}$$

which implies that q_2 is in fact in $B^*_{\frac{1}{2}}(\mathbb{R}^d)$.

Step 3: The proof of (4.23).

It remains to prove that $q_2=0$. We shall first establish an appropriate representation formula for (q_2,ψ) and then prove that $(q_2,\psi)=0$, for any $\psi\in C_0^\infty(\mathbb{R}^d)$. This will finish the proof of (4.23).

Firstly from (4.10) and (4.11), q_2^ε satisfies the equation

$$i\varepsilon\alpha_\varepsilon q_2^\varepsilon+\triangle q_2^\varepsilon+n_{z_1}^2(0)q_2^\varepsilon=\tilde{n}_{z_1,\varepsilon}\overline{w}^\varepsilon_{z_1}(y). \qquad (4.52)$$

Let $\chi(r) \in C_0^\infty(0, \infty)$ be a cut-off function such that

$$\chi(r) = \begin{cases} 1, & |r| \leqslant 2, \\ 0, & |r| \geqslant 3. \end{cases}$$

Then multiplying (4.52) by $\chi_L(y) = \chi(\frac{|y|}{L})$, we find

$$\begin{aligned} &i\varepsilon\alpha_\varepsilon\chi_L q_2^\varepsilon + \triangle(\chi_L q_2^\varepsilon) + n_{z_1}^2(0)\chi_L q_2^\varepsilon \\ &\quad = \tilde{n}_{z_1,\varepsilon}\overline{w}_{z_1}^\varepsilon(y)\chi_L - \triangle\chi_L q_2^\varepsilon + 2\mathrm{div}(\nabla\chi_L q_2^\varepsilon). \end{aligned} \tag{4.53}$$

For any test function $\psi \in C_0^\infty(\mathbb{R}^d)$, we take L sufficiently large such that $\mathrm{supp}\,\psi \subset \{y \in \mathbb{R}^d : |y| \leqslant L\}$. It follows that

$$\begin{aligned} (q_2^\varepsilon, \psi) &= (\chi_L q_2^\varepsilon, \psi) \\ &= -(R_0(i\kappa)(\tilde{n}_{z_1,\varepsilon}\overline{w}_{z_1}^\varepsilon\chi_L), \psi) - (R_0(i\kappa)(-\triangle\chi_L q_2^\varepsilon + 2\,\mathrm{div}(\nabla\chi_L q_2^\varepsilon)), \psi). \end{aligned} \tag{4.54}$$

Furthermore, by (4.24), we have

$$(R_0(i\kappa)(\tilde{n}_{z_1,\varepsilon}\overline{w}_{z_1}^\varepsilon\chi_L), \psi) = (\tilde{n}_{z_1,\varepsilon}\overline{w}_{z_1}^\varepsilon\chi_L, R_0(-i\kappa)\psi). \tag{4.55}$$

Since $\psi \in C_0^\infty(R^d)$, we have [1],

$$D^\alpha R_0(-i\kappa)\psi = R_0(-i\kappa)(D^\alpha\psi) \to R_0(-i0)(D^\alpha\psi) \quad \text{in} \quad B_{\frac{1}{2}}^*(\mathbb{R}^d). \tag{4.56}$$

Note that by (3.20) and (4.12), for any $s > \frac{p}{2} + 2$, there holds

$$\|\overline{w}_{z_1}^\varepsilon\|_{L^{2,-s}(\mathbb{R}^d)} \leqslant \|q_1^\varepsilon\|_{L^{2,-s}(\mathbb{R}^d)} + \|w_{z_1}^\varepsilon\|_{L^{2,-s}(\mathbb{R}^d)} \leqslant C_{z_1}. \tag{4.57}$$

Combining (4.57) with (4.56), and taking $\varepsilon \to 0$ in (4.55), we obtain

$$(R_0(i\kappa)(\tilde{n}_{z_1,\varepsilon}\bar{w}_{z_1}^\varepsilon\chi_L), \psi) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.58}$$

While by summing up (4.51) and (4.56), we arrive at

$$\begin{aligned} &(R_0(i\kappa)(-\triangle\chi_L q_2^\varepsilon + 2\,\mathrm{div}(\nabla\chi_L q_2^\varepsilon)), \psi) \\ &\quad = -(\triangle\chi_L q_2^\varepsilon, R_0(-i\kappa)\psi) - 2(\nabla\chi_L q_2^\varepsilon, \nabla(R_0(-i\kappa)\psi)) \\ &\quad \to -(\triangle\chi_L q_2, R_0(-i0)\psi) - 2(\nabla\chi_L q_2, \nabla(R_0(-i0)\psi)) \end{aligned} \tag{4.59}$$

as $\varepsilon \to 0$.

Summing up (4.58)–(4.59), and taking $\varepsilon \to 0$ in (4.54), we arrive at

$$(q_2, \psi) = (\triangle\chi_L q_2, R_0(-i0)\psi) + 2(\nabla\chi_L q_2, \nabla(R_0(-i0)\psi)). \tag{4.60}$$

In the derivation of (4.60), we used the fact that for each fixed L, the multiplier $\nabla\chi_L = \frac{1}{L}\nabla(\chi)(\frac{\cdot}{L})$ is bounded from $B^*_{\frac{1}{2}}(\mathbb{R}^d)$ to $B_{\frac{1}{2}}(\mathbb{R}^d)$ which is trivial because $\nabla\chi_L$ is of compact support. This boundedness is uniform in $L \geqslant 1$. In fact, let $k_0 \in \mathbb{N}$ such that $2^{k_0-1} \leqslant L < 2^{k_0}$. Then for $v \in B^*_{\frac{1}{2}}(\mathbb{R}^d)$,

$$\begin{aligned}\frac{1}{L}\,\|(\nabla\chi)\left(\frac{\cdot}{L}\right)v\|_{B_{\frac{1}{2}}(\mathbb{R}^d)} &\leqslant \frac{C}{L}\sum_{k_0 \leqslant j \leqslant k_0+1} 2^{\frac{j}{2}}\|v\|_{L^2(2^j<|x|\leqslant 2^{j+1})}\\ &\leqslant C_1 \sup_{1\leqslant R\leqslant 2^{k_0+2}} \frac{1}{R^{\frac{1}{2}}}\|v\|_{L^2(B(0,R))} \leqslant C_1\|v\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)}\end{aligned}$$

uniformly in L. It is easy to see that the multiplier $\triangle\chi_L$ is also uniformly bounded from $B^*_{\frac{1}{2}}(\mathbb{R}^d)$ to $B_{\frac{1}{2}}(\mathbb{R}^d)$. Since $q_2 \in B^*_{\frac{1}{2}}(\mathbb{R}^d)$, we can take a sequence of $q_{2,n} \in C_0^\infty(\mathbb{R}^d)$ with $q_{2,n} \to q_2$ in $B^*_{\frac{1}{2}}(\mathbb{R}^d)$. It follows from (4.60) that

$$\begin{aligned}|(q_2, \psi)| \leqslant{}& C\|q_2 - q_{2,n}\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)}(\|R_0(-i0)\psi\|_{B_{\frac{1}{2}}{}^*(\mathbb{R}^d)} + \|\nabla R_0(-i0)\psi\|_{B^*_{\frac{1}{2}}(\mathbb{R}^d)})\\ &+|(\triangle\chi_L q_{2,n}, R_0(-i0)\psi)| + 2|(\nabla\chi_L q_{2,n}, \nabla(R_0(-i0)\psi))|,\end{aligned} \tag{4.61}$$

where C is independent of L and n. Since $\operatorname{supp}\nabla\chi_L$ and $\operatorname{supp}\Delta\chi_L$ are contained in $\{x; |x| \geqslant 2L\}$ and $q_{2,n}$ is of compact support, taking first the limit $L \to \infty$ and then the limit $n \to \infty$ in (4.61), we obtain $(q_2, \psi) = 0$. This completes the proof of the Lemma. □

Note that the assumption (A.6) with some $\delta_0 > 0$ uniformly in y and z_1 implies (A.4). For simplicity, we assume that (A.6) is satisfied in this uniform version. Therefore, combining Lemma 4.2 with Lemma 4.3, we conclude

Theorem 4.5. *We assume that $n(x)$ and $\phi(y)$ satisfy* (1.3), (1.4), (A.5) *and* (A.6), *and that $S(y)$ decays sufficiently fast at infinity. Let $w^\varepsilon_{z_1}(y)$, $w_{z_1}(y)$ be the solutions to* (4.9) *and* (4.10), *respectively. Then for any $s > \frac{p}{2} + 2$, there is a subsequence of $\{w^\varepsilon_{z_1}\}$, which we still denote it by $\{w^\varepsilon_{z_1}\}$, such that*

$$w^\varepsilon_{z_1} \rightharpoonup w_{z_1} \quad \textit{weakly in} \quad L^{2,-s}(\mathbb{R}^d) \tag{4.62}$$

as $\varepsilon \to 0$.

In [4], F. Castella proves the similar result as that of Theorem 4.5 for the source term supported near one point by using a time-dependent approach. Theorem 4.5 is the key step to calculate the source term Q in the Liouville equation

$$\alpha f + \xi \cdot \nabla_x f + \frac{1}{2} \nabla_x n^2(x) \cdot \nabla_\xi f = Q(x, \xi), \tag{4.63}$$

which is satisfied by the semi-classical measure f given by Theorem 4.1.

5. The limit to the Liouville equation

In this section, we calculate the equation satisfied by the semi-classical measure f given by Theorem 4.1 and prove its radiation property in a weak form.

Theorem 5.1. *Let A and S be smooth enough functions with A of compact support and S decaying sufficiently fast at infinity, and for some $N_0 > d + 4$, there holds*

$$\sup_{x \in \mathbb{R}^d} \left| |x|^{N_0} \nabla n^2(x) \right| < \infty. \tag{5.1}$$

Then under the assumptions of Theorems 4.1 and 4.5, there holds

(i) *Assume that $\lim_{\varepsilon \to 0} \alpha_\varepsilon = \alpha > 0$. Then the semi-classical measure $f(x, \xi)$ constructed in Theorem 4.1 satisfies the Liouville equation*

$$\alpha f + \xi \cdot \nabla_x f + \frac{1}{2} \nabla_x n^2(x) \cdot \nabla_\xi f = Q(x, \xi) \tag{5.2}$$

with

$$\begin{aligned} Q(x, \xi) = 2^p \pi^{p+1} |\hat{S}(\xi)|^2 \int_{R^p} & dz_1 \delta(x - (z_1, 0)) \delta(\xi_1 - \nabla \phi(z_1)) \\ & \times \delta(n(z_1, 0)^2 - |\xi|^2) |A(z_1)|^2. \end{aligned} \tag{5.3}$$

Moreover, the Sommerfeld radiation condition holds in the following weak form. Let

$$\Omega = \{(x, \xi) \in \mathbb{R}^{2d};\ |\xi^2 - n^2(x)| < \delta\}$$

for some $\delta > 0$ small enough. For any $R \in \mathcal{D}(\Omega)$, let

$$g(x, \xi) = \int_0^\infty e^{-\alpha s} R(X^s(x, \xi), \Xi^s(x, \xi))\, ds \tag{5.4}$$

with $(X^s(x,\xi), \Xi^s(x,\xi))$ *defined by* (1.7) *for* $(x,\xi) \in \Omega$, *so that* $g(x,\xi)$ *solves the dual equation to* (5.2)

$$\alpha g - \xi \cdot \nabla_x g - \frac{1}{2} \nabla_x n^2(x) \cdot \nabla_\xi g = R. \tag{5.5}$$

Then we have the following duality property:

$$\int_{R^{2d}} \overline{R(x,\xi)} f(x,\xi)\, dx\, d\xi = \int_{\mathbb{R}^{2d}} \overline{g(x,\xi)} Q(x,\xi)\, dx\, d\xi, \quad \forall R \in \mathcal{D}(\Omega). \tag{5.6}$$

(ii) *If* $\alpha = 0$, *assume further that* $\alpha_\varepsilon \geqslant \varepsilon^\gamma$ *for* $0 < \gamma < 1$. *Then* (5.2) *and* (5.6) *still hold with* α *in* (5.2) *and* (5.5) *replaced by* $+0$.

Remark 5.2. Note that Ω is invariant by the solutions to (1.7) and that (A.5) with $z_1 = 0$ implies that the classical flow is non-trapping for initial data $(x,\xi) \in \Omega$. One can show that g is a well-defined smooth function on Ω and satisfies (5.5). In fact, since R is of compact support, for (x,ξ) in $\operatorname{supp} g$,

$$g(x,\xi) = \int_{s_0}^{s_0+T_1} e^{-\alpha s} R(X^s(x,\xi), \Xi^s(x,\xi))\, ds,$$

where $s_0 \geqslant 0$, which depends continuously on (x,ξ), is the time needed for the trajectory issuing from (x,ξ) to enter into the support of R, and T_1 the time needed for it to leave $\operatorname{supp} R$. The non-trapping condition ensures that T_1 is finite and only depends on $\operatorname{supp} R$. See the details below. Since Q is a distribution with compact support, the right-hand side of (5.6) makes sense. The relation (5.6) for all $R \in \mathcal{D}(\Omega)$ shows that f is defined in sense of distribution on Ω by

$$f(x,\xi) = \int_0^\infty e^{-\alpha s} Q(X^{-s}(x,\xi), \Xi^{-s}(x,\xi))\, ds. \tag{5.7}$$

It can be checked that f is a weak solution of (5.2). This solution has the following weak radiation property:

$$\lim_{t \to -\infty} f \circ \Phi^t = 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{5.8}$$

where $\Phi^t(x,\xi) = (X^t(x,\xi), \Xi^t(x,\xi))$. In fact, for $R \in \mathcal{D}(\Omega)$,

$$\int_{R^{2d}} \overline{R(x,\xi)} f(\Phi^t(x,\xi))\, dx\, d\xi = \int_{R^{2d}} \int_0^\infty e^{-\alpha s} \overline{R(\Phi^{s-t}(x,\xi))}\, ds\, Q(x,\xi)\, dx\, d\xi. \tag{5.9}$$

Since $\operatorname{supp} Q$ is compact, the non-trapping condition implies that there exists $T_1 > 0$ such that for all $(x, \xi) \in \operatorname{supp} Q$, one has

$$|\Phi^\tau(x, \xi)| > R_1, \quad \forall \tau > T_1,$$

where R_1 is taken large enough so that $\operatorname{supp} R \subset \{|x| + |\xi| < R_1\}$. This shows

$$\int_{R^{2d}} \int_0^\infty e^{-\alpha s} \overline{R(\Phi^{s-t}(x, \xi))}\, ds\, Q(x, \xi)\, dx\, d\xi = 0, \quad t < -T_1.$$

This proves the weak radiation property (5.8) of the limiting semi-classical measure f. See [4] for another approach to the radiation properties of $w^\varepsilon_{z_1}$ for point source case.

Proof of Theorem 5.1. As the proof of (5.2) is similar with and much easier than that of (5.6), we only present the proof of (5.6) here. If $\alpha > 0$, we can use (5.4) and Lemma 5.3 directly to prove (5.6). The difficulty lies in the case when $\alpha = 0$. Actually the proof of the case with $\alpha = 0$ will imply the Theorem for the case when $\alpha > 0$. For $\alpha = 0$, motivated by Castella et al. [5], we will use an approximate argument to the test function here. Let $(X^s(x, \xi), \Xi^s(x, \xi))$ be the solution to (1.7) for $(x, \xi) \in \Omega$, we define

$$g_\varepsilon(x, \xi) = \int_0^\infty e^{-\alpha_\varepsilon s} R(X^s(x, \xi), \Xi^s(x, \xi))\, ds, \tag{5.10}$$

which solves the following approximate dual equation:

$$\alpha_\varepsilon g - \xi \cdot \nabla_x g - \tfrac{1}{2} \nabla_x n^2 \cdot \nabla_\xi g = R(x, \xi). \tag{5.11}$$

In the sequence, we will denote $R(X^s(x, \xi), \Xi^s(x, \xi))$ by $R^s(x, \xi)$ for convenience.

Then by (4.4) and integration by parts, we have

$$(f_\varepsilon, R) = -(\Theta^\varepsilon[n^2] f_\varepsilon,\ g_\varepsilon) - (f_\varepsilon,\ \tfrac{1}{2} \nabla_x n^2 \cdot \nabla_\xi g_\varepsilon) + (Q_\varepsilon, g_\varepsilon). \tag{5.12}$$

Comparing (5.12) with (5.6), we achieve (5.6) provided that we have

$$(\Theta^\varepsilon[n^2] f_\varepsilon,\ g_\varepsilon) \to -\tfrac{1}{2}(f,\ \nabla_x n^2 \cdot \nabla_\xi g), \tag{5.13}$$

$$(f_\varepsilon,\ \nabla_x n^2 \cdot \nabla_\xi g_\varepsilon) \to (f,\ \nabla_x n^2 \cdot \nabla_\xi g), \tag{5.14}$$

$$(Q_\varepsilon,\ g_\varepsilon) \to (Q,\ g) \tag{5.15}$$

as $\varepsilon \to 0$.

The proof of (5.13)–(5.15) follows the ideas of Benamou et al. [2] and Castella et al. [5], and is, however technically much more involved here. The main difficulty is the global control of g which we prove in Lemma 5.3 below, where the condition (A.5) plays once more an important role. In the sequel, we will divide the proof into two steps.

Step 1: The proof of (5.13) and (5.14).

Firstly by (4.5), we have

$$(\Theta^{\varepsilon}[n^2]f_{\varepsilon}, g_{\varepsilon}) = \int_{\mathbb{R}^{2d}_{x,\eta}} f_{\varepsilon}(x,\eta)\frac{1}{(2\pi)^{\frac{d}{2}}} \times \int_{R^d_y} e^{i\eta y}\frac{n^2(x+\frac{\varepsilon y}{2}) - n^2(x-\frac{\varepsilon y}{2})}{2i\varepsilon}\widehat{g_{\varepsilon}}(x,y)\,dy\,dx\,d\eta, \quad (5.16)$$

where $\widehat{g_{\varepsilon}}(x,y) = \mathcal{F}_{\xi\to y}(g_{\varepsilon}(x,\xi))$.

Denote

$$G_{\varepsilon}(x,\eta) = \frac{1}{2i(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d_y} e^{iy\eta}\int_{-1}^{1}\nabla n^2\left(x+\frac{\varepsilon\theta y}{2}\right)d\theta y\widehat{g_{\varepsilon}}(x,y)\,dy.$$

Then by Theorem 4.1, to prove (5.13), we only need to show that

$$G_{\varepsilon}(x,\eta) \to -\tfrac{1}{2}\nabla n^2(x)\cdot\nabla_{\eta}g(x,\eta) \quad \text{in} \quad X_{1+0}. \quad (5.17)$$

Here $1+0$ means $1+\eta$ for any $\eta > 0$ and the constant C appeared below depends on η. In order to prove (5.17), we decompose the X_{1+0} norm of the difference as

$$\begin{aligned}
&\int_{R^d_y}\sup_{x\in R^d}\left\{\langle x,y\rangle^{1+0}\left|\frac{1}{2}\int_{-1}^{1}\nabla n^2\left(x+\frac{\varepsilon\theta y}{2}\right)d\theta y\widehat{g_{\varepsilon}}(x,y) - \nabla n^2(x)y\hat{g}(x,y)\right|\right\}dy\\
&\leqslant \int_{R^d_y}\sup_{x\in R^d}\left\{\langle x,y\rangle^{1+0}\left|\frac{1}{2}\int_{-1}^{1}\left(\nabla n^2\left(x+\frac{\varepsilon\theta y}{2}\right) - \nabla n^2(x)\right)d\theta y\widehat{g_{\varepsilon}}(x,y)\right|\right\}dy\\
&\quad + \int_{R^d_y}\sup_{x\in R^d}\left\{\langle x,y\rangle^{1+0}|\nabla n^2(x)y(\widehat{g_{\varepsilon}}(x,y) - \hat{g}(x,y))|\right\}dy\\
&= I + II. \qquad (5.18)
\end{aligned}$$

Let M, N be positive numbers, which will be chosen later. Then by (5.39), we can estimate I by the following:

$$\begin{aligned}
|I| \leqslant C\Bigg\{&\int_{|y|\leqslant\varepsilon^{-1+0}}\sup_{x\in R^d}\left\{\langle x,y\rangle^{1+0}\int_{-1}^{1}\langle x\rangle^{N}\left|\nabla n^2\left(x+\frac{\varepsilon\theta y}{2}\right)\right.\right.\\
&\left.\left.-\nabla n^2(x)\right|d\theta|y|\frac{\langle x\rangle^{M-N}}{\langle y\rangle^{M}}\right\}dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{|y|\geqslant \varepsilon^{-1+0}} \sup_{|x|\geqslant \varepsilon|y|} \left\{ \langle x, y\rangle^{1+0} \int_{-1}^{1} \langle x\rangle^N \left(\left| \nabla n^2 \left(x + \frac{\varepsilon\theta y}{2} \right) \right| \right.\right. \\
&\left.\left. + |\nabla n^2(x)| \right) d\theta |y| \frac{\langle x\rangle^{M-N}}{\langle y\rangle^M} \right\} dy \\
&\left. + \int_{|y|\geqslant \varepsilon^{-1+0}} \sup_{|x|\leqslant \varepsilon|y|} \left\{ \langle x, y\rangle^{1+0} \|\nabla n^2\|_{L^\infty} |y| \frac{\langle x\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M} \right\} dy \right\} \\
&\triangleq I_1 + I_2 + I_3,
\end{aligned}
\tag{5.19}
$$

where $a \wedge b = \min\{a, b\}$. Firstly, by the assumptions in the Theorem, we can choose M, N so that $M > d+2$, $N_0 > N > M+1$, then (5.1) implies

$$
|I_1| \leqslant C \sup_{x\in R^d} \left(\langle x\rangle^N |\nabla n(x+o(1))^2 - \nabla n^2(x)| \right) \to 0 \tag{5.20}
$$

as $\varepsilon \to 0$. Secondly with M, N chosen as above, when $|x| \geqslant \varepsilon|y|$, $|x + \frac{\varepsilon\theta y}{2}| \geqslant \frac{|x|}{2}$, there holds

$$
|I_2| \leqslant C \sup_{x\in R^d} \left(\langle x\rangle^N |\nabla n^2(x)| \right) \int_{|y|\geqslant \varepsilon^{-1+0}} \frac{1}{\langle y\rangle^{M-2-0}} dy \to 0. \tag{5.21}
$$

And finally

$$
\begin{aligned}
|I_3| &\leqslant \|\nabla n^2\|_{L^\infty} \int_{|y|\geqslant \varepsilon^{-1+0}} \frac{\langle \varepsilon y\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^{M-2-0}} dy \\
&\leqslant \|\nabla n^2\|_{L^\infty} \varepsilon^{M-2-d-0} \left(\int_{\mathbb{R}^d} \frac{\langle z\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle z\rangle^{M-2-0}} dz \right),
\end{aligned}
$$

note by the assumptions that $\alpha \geqslant \varepsilon^\gamma$, $\gamma < 1$

$$
\int_{\mathbb{R}^d} \frac{\langle z\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle z\rangle^{M-2-0}} dz \leqslant \varepsilon^{-M\gamma} \int_{\mathbb{R}^d} \frac{1}{\langle z\rangle^{M-2-0}} dz,
$$

we arrive at

$$
|I_3| \leqslant C\varepsilon^{M-2-d-0-M\gamma} \to 0 \tag{5.22}
$$

if we take $M > \frac{2+d}{1-\gamma}$. Then by summing up (5.19)–(5.22), we obtain

$$
|I| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{5.23}
$$

On the other hand, by a similar proof of (5.39), we get

$$\begin{aligned}\langle y\rangle^M|\widehat{g_\varepsilon}(x,y)-\hat{g}(x,y)| &\leqslant C\langle x\rangle^M(1-\exp(-\alpha_\varepsilon|x|))\\ &\leqslant C\alpha_\varepsilon\langle x\rangle^{M+1},\end{aligned}\tag{5.24}$$

by taking $N_0-2>M>d+2$ in (5.24), we obtain

$$|II|\leqslant\alpha_\varepsilon\int_{R^d}\sup_x\left(\langle x\rangle^{M+2+0}|\nabla n^2(x)|\right)\langle y\rangle^{-M+2+0}\,dy\to 0\tag{5.25}$$

as $\varepsilon\to 0$.

By summing up (5.18), (5.23) and (5.25), to prove (5.17), we still need to prove that $\nabla n^2(x)\nabla_\eta g(x,\eta)$ is in X_{1+0}. In fact, by taking $N_0-1>M>d+2$ and using (5.39), we have

$$\begin{aligned}&\frac{1}{2}\int_{R^d_y}\sup_x\left\{\langle x,y\rangle^{1+0}|\nabla n^2(x)||y|\hat{g}(x,y)|\right\}dy\\ &\quad\leqslant C\ \sup_x\{\langle x\rangle^{M+1+0}|\nabla n^2(x)|\}\int_{R^d}\frac{1}{\langle y\rangle^{M-2-0}}\,dy\leqslant C.\end{aligned}\tag{5.26}$$

which proves (5.13). Then by a similar but easier proof of (5.13), we can also prove (5.14).

Step 2: The proof of (5.15).

With (5.40), let us modify the proof of (119) in [5] to prove (5.15). Firstly, by (121) of Castella et al. [5], we obtain

$$\langle Q_\varepsilon,g_\varepsilon\rangle=-\mathrm{Im}\left(\int_{\mathbb{R}^{2d+p}}A(z_1)S(x+y)\overline{w^\varepsilon_{z_1}(x)}\hat{g}_\varepsilon\left(z_1+\varepsilon\left(x+\frac{y}{2}\right),y\right)dx\,dy\,dz_1\right),$$

which can be decomposed as

$$\begin{aligned}&\langle Q_\varepsilon,g_\varepsilon\rangle\\ &\quad=-\mathrm{Im}\left(\int_{\mathbb{R}^{2d+p}}A(z_1)S(x+y)\overline{w^\varepsilon_{z_1}(x)}\left(\hat{g}_\varepsilon\left(z_1+\varepsilon\left(x+\frac{y}{2}\right),y\right)\right.\right.\\ &\qquad\left.\left.-\hat{g}_\varepsilon(z_1,y)\right)dx\,dy\,dz_1\right)\\ &\qquad-\mathrm{Im}\left(\int_{\mathbb{R}^{2d+p}}A(z_1)S(x+y)\overline{w^\varepsilon_{z_1}(x)}(\hat{g}_\varepsilon(z_1,y)-\hat{g}(z_1,y))\,dx\,dy\,dz_1\right)\end{aligned}$$

$$-\mathrm{Im}\left(\int_{\mathbb{R}^{2d+p}} A(z_1)S(x+y)\overline{w^{\varepsilon}_{z_1}(x)}\hat{g}(z_1,y)\,dx\,dy\,dz_1\right)$$

$$\triangleq V_1^{\varepsilon}+V_2^{\varepsilon}+V_3^{\varepsilon}. \tag{5.27}$$

Next let us estimate the above terms separately. Firstly as $A(z_1)$ has compact, let $A_0>0$ large enough such that $|z_1|\leqslant A_0$ for all $z_1\in \operatorname{supp} A$. Then by (3.20), and taking $N>\frac{p}{2}+1$, we have

$$\begin{aligned}
&\sup_{|z_1|\leqslant A_0}\left|\int_{\mathbb{R}^d} S(x+y)\overline{w^{\varepsilon}_{z_1}(x)}\hat{g}(z_1,y)\,dx\right| \\
&\quad\leqslant \sup_{|z_1|\leqslant A_0}\int_{\mathbb{R}^d}\langle x+y\rangle^N|S(x+y)|\,\frac{|\overline{w^{\varepsilon}_{z_1}(x)}|}{\langle x\rangle^{\frac{p}{2}+1+0}}\,\frac{\langle x\rangle^{\frac{p}{2}+1+0}}{\langle x+y\rangle^N}|\hat{g}(z_1,y)|\,dx \\
&\quad\leqslant C\|S\|_{L^{2,N}}\sup_{\substack{|z_1|\leqslant A_0\\ x\in\mathbb{R}^d}}\left(\frac{\langle |x|+|y|\rangle^{\frac{p}{2}+1+0}}{\langle x\rangle^N}\,|\hat{g}(z_1,y)|\right),
\end{aligned} \tag{5.28}$$

which together (5.39) implies that

$$\sup_{|z_1|\leqslant A_0}\left|\int_{\mathbb{R}^d} S(x+y)\overline{w^{\varepsilon}_{z_1}(x)}\hat{g}(z_1,y)\,dx\right|\leqslant \frac{C}{\langle y\rangle^{M-\frac{p}{2}-1-0}}. \tag{5.29}$$

As for any fixed y, $S(x+y)\in L^{2,\frac{p}{2}+2+0}(\mathbb{R}^d)$, by (4.62), (5.29) and Lebesgue dominated convergence Theorem, we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} V_3^{\varepsilon} &= -\mathrm{Im}\left(\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{d+p}} A(z_1)(S(\cdot+y),\ w^{\varepsilon}_{z_1}(\cdot))\hat{g}(z_1,y)\,dy\,dz_1\right) \\
&= -\mathrm{Im}\left(\int_{\mathbb{R}^{d+p}} A(z_1)(S(\cdot+y),\ w_{z_1}(\cdot))\hat{g}(z_1,y)\,dy\,dz_1\right) = (Q,g),
\end{aligned} \tag{5.30}$$

where we denote $(\cdot,\cdot)$ the L^2 inner product of the two functions, and in the next to last step of the above, we used (34) of Castella et al. [5].

Exactly similarly to the proof of (5.28), we get

$$|V_2^{\varepsilon}|\leqslant C\int_{\mathbb{R}^d}\sup_{\substack{|z_1|\leqslant A_0\\ x\in\mathbb{R}^d}}\left(\frac{\langle |x|+|y|\rangle^{\frac{p}{2}+1+0}}{\langle x\rangle^N}\,|\widehat{g_\varepsilon}(z_1,y)-\hat{g}(z_1,y)|\right)dy, \tag{5.31}$$

which together with (5.24) implies that

$$|V_2^{\varepsilon}|\leqslant C\alpha_\varepsilon\int_{\mathbb{R}^d}\frac{1}{\langle y\rangle^{M-\frac{p}{2}-1-0}}\,dy\to 0, \tag{5.32}$$

as $\varepsilon \to 0$. Finally the proof of (5.28) also yields

$$|V_1^\varepsilon| \leqslant C \int_{\mathbb{R}^d} \sup_{\substack{|z_1|\leqslant A_0\\ x\in\mathbb{R}^d}} \times \left(\frac{\langle |x|+|y|\rangle^{\frac{p}{2}+1+0}}{\langle x\rangle^N} \left|\widehat{g_\varepsilon}\left(z_1+\varepsilon\left(x+\frac{y}{2}\right), y\right) - \widehat{g_\varepsilon}(z_1,y)\right| \right) dy. \tag{5.33}$$

As in the proof of (5.19) and (139) of Castella et al. [5], we naturally split the domain as

$$D_1 := \left\{(x,y):\ \left|x+\frac{y}{2}\right| \leqslant \varepsilon^{-1+0}\right\},\ D_2 := \left\{(x,y): \left|x+\frac{y}{2}\right| \geqslant \varepsilon^{-1+0}, |x| \geqslant \frac{|y|}{4}\right\},$$
$$D_3 := \left\{(x,y): \left|x+\frac{y}{2}\right| \geqslant \varepsilon^{-1+0}, |x| \leqslant \frac{|y|}{4}\right\}. \tag{5.34}$$

On the set D_1, by (5.40), there holds

$$(5.33)_{D_1} \leqslant C \int_{\mathbb{R}^d} \frac{o(1)}{\langle y\rangle^{M-\frac{p}{2}-1-0}}\, dy. \tag{5.35}$$

Let $1_{D_2}(x,y)$ be the characteristic function on the set D_2. Then by (5.39), we have

$$\begin{aligned}(5.33)_{D_2} &\leqslant C \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} \left(\frac{\langle |x|+|y|\rangle^{\frac{p}{2}+1+0}}{\langle x\rangle^N} \frac{(\varepsilon(|x|+|y|))^M \wedge \alpha_\varepsilon^{-M}}{|y|^M} 1_{D_2}(x,y) \right) dy \\ &\leqslant C \int_{\mathbb{R}^d} \frac{1}{\langle y\rangle^M}\, dy \sup_{|x|\geqslant C\varepsilon^{-1+0}} \left(\langle x\rangle^{\frac{p}{2}+1+0-N} (\langle \varepsilon x\rangle^M \wedge \alpha_\varepsilon^{-M}) \right) \\ &\leqslant C\varepsilon^{(N-\frac{p}{2}-1-0)-M\gamma} \to 0 \end{aligned} \tag{5.36}$$

if we take $N > \frac{p}{2}+1+M\gamma$ and $M > d$. The proof of (5.36) implies the following estimate for (5.33) on the set D_3

$$\begin{aligned}(5.33)_{D_3} &\leqslant C \int_{|y|\geqslant C\varepsilon^{-1+0}} |y|^{\frac{p}{2}+1+0} \frac{\langle \varepsilon y\rangle^M \wedge \alpha_\varepsilon^{-M}}{\langle y\rangle^M}\, dy \\ &\leqslant C\varepsilon^{M-(1+\gamma)(\frac{p}{2}+1+d+0)} \to 0 \end{aligned} \tag{5.37}$$

if we take M large enough such that $M > (1+\gamma)(\frac{p}{2}+1+d)$. Therefore under the assumptions in (ii) of the Theorem, by summing up (5.33)–(5.37), we arrive at

$$\lim_{\varepsilon\to 0} |V_1^\varepsilon| = 0. \tag{5.38}$$

Summing up (5.27), (5.30), (5.32) and (5.38) together, we achieve (5.15).
Combining step 1 with step 2, we obtain (5.6). □

To complete the proof Theorem 5.1, we still need the following technical lemma.

Lemma 5.3. *Assume* (1.4) *with* $\rho_0 > 0$ *and* (A.5) *with* $z_1 = 0$. *Then for* g *defined by* (5.4), *there holds for any* $M > 0$,

$$\langle y\rangle^M |\widehat{g}(x,y)| \leqslant C\langle x\rangle^M \exp(-\alpha|x|) \tag{5.39}$$

and

$$\langle y\rangle^M |\nabla_x \widehat{g}(x,y)| \leqslant C\langle x\rangle^M \exp(-\alpha|x|). \tag{5.40}$$

where both the constants C *in* (5.39) *and* (5.40) *are independent of* $\alpha \geqslant 0$.

Remark 5.4. Eq. (5.39) seemingly looks like (125) for constant refraction index in [5]. However the proof here is much more complicated than that there. To prove Lemma 5.3, we need to study the properties of the flow (X^s, Ξ^s) globally in s, x, ξ. Note that the condition (A.5) implies, among others, that the sojourn time of the flow in a ball of radius R is always proportional to R for any $R > 0$ and in any part of the space. This property may fail under the general non-trapping assumption (2.3) on X^s.

Proof. To estimate $|y|^M\ |\widehat{g}(x,y)|$ for $M \geqslant 1$, we need to estimate $(\nabla_\xi^\beta X^s, \nabla_\xi^\beta \Xi^s)$ for $1 \leqslant |\beta| \leqslant M$ uniformly in (x,ξ) such that $(X^s, \Xi^s) \in \operatorname{supp} R$ for some $s \geqslant 0$. Again for a clearer presentation, we divide the proof into several steps.

Step 1: The estimate of (5.39) for $M = 0$.

As $\operatorname{supp} R \subset \Omega$ and is compact, let $R_0 > 0$ be such that

$$|x| + |\xi| < R_0, \quad \text{for} \quad (x,\xi) \in \operatorname{supp} R. \tag{5.41}$$

Notice that the classical trajectories of (1.7) leave Ω invariant. Therefore $g(x,\xi)$ defined by (5.4) has compact support in the ξ variables, and

$$|\Xi^s(x,\xi)| \leqslant R_0 \quad \text{and} \quad \big||\Xi^s(x,\xi)|^2 - n^2(X^s(x,\xi))\big| < \delta. \tag{5.42}$$

Then by (1.7) and (5.42), a direct computation shows that

$$\begin{aligned}\frac{d^2}{ds^2}|X^s|^2 &= 2\frac{d}{ds}(X^s \cdot \Xi^s) = 2|\Xi^s|^2 + X^s \cdot \nabla n^2(X^s)\\ &= 2(|\Xi^s|^2 - n^2(X^s)) + 2n^2(X^s) + X^s \cdot \nabla n^2(X^s)\end{aligned}$$

which together with the non-trapping condition (A.5) with $z_1 = 0$ implies that

$$\frac{d^2}{ds^2}|X^s|^2 \geqslant 2\delta_1 - 2\delta \geqslant \tilde{\delta}_1 > 0 \tag{5.43}$$

if $\delta > 0$ is small enough. Integrating the above inequality twice, we find

$$|X^s|^2 \geqslant \frac{\tilde{\delta}_1}{2} s^2 + |x|^2 + 2sx \cdot \xi \geqslant \frac{\tilde{\delta}_1}{2} s^2 + |x|^2 - 2s\delta_2|x|, \tag{5.44}$$

where we denote $\delta_2 = \sqrt{\max|n|^2 + \delta}$, as $||\xi|^2 - n^2(x)| \leqslant \delta$ for $(x, \xi) \in \Omega$. Then if (X^s, Ξ^s) belongs to $\operatorname{supp} R$, we can find three positive constants c_1, c_2 and c_3 such that

$$c_1|x| - c_2 \leqslant s \leqslant c_3|x| + c_2. \tag{5.45}$$

On the other hand, for any $(x, \xi) \in \Omega$, we denote $s_0(x, \xi)$, in short by s_0, the smallest $s \geqslant 0$ such that (X^s, Ξ^s) belongs to $\operatorname{supp} R$, which satisfies (5.41). Then by (5.45), we get

$$c_1|x| - c_2 \leqslant s_0 \leqslant c_3|x| + c_2. \tag{5.46}$$

While by (5.44), if $|x| \leqslant R_0$, we can find some positive constant T_1, such that

$$|X^s(x, \xi)| \geqslant 2R_0, \quad \forall s > T_1. \tag{5.47}$$

Therefore, for $s > T_1 + s_0$, $R^s(x, \xi) = R^{s-s_0}(X^{s_0}(x, \xi), \Xi^{s_0}(x, \xi)) = 0$, and the integration for s in (5.4) is in fact taken over $[s_0, s_0 + T_1]$, where T_1 depends only on the support of R. From (5.4) and (5.46), we obtain

$$\begin{aligned} |\widehat{g}(x, y)| &= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \int_{|\xi| \leqslant R_0} \int_{s_0}^{s_0+T_1} e^{-iy\xi} \exp(-\alpha s) R^s(x, \xi)\, ds\, d\xi \right| \\ &\leqslant C\|R\|_{L^\infty} \int_{s_0}^{s_0+T_1} \exp(-\alpha s)\, ds \leqslant C\|R\|_{L^\infty} \exp(-\alpha|x|), \end{aligned} \tag{5.48}$$

which proves (5.39) for $M = 0$.

Eq. (5.45) shows that for $(X^s, \Xi^s) \in \operatorname{supp} R$, s, s_0 and $\langle x \rangle$ are all equivalent. Let $(x, \xi) \in \Omega$ such that $(X^s(x, \xi), \Xi^s(x, \xi))$ belongs to $\operatorname{supp} R$ for some $s > 0$. Let $s_1 \geqslant 0$ such that $|X^{s_1}| + |\Xi^{s_1}| < 2R_0$. s_1 can be chosen locally independent of (x, ξ) when (x, ξ) varies in a small neighborhood.

Step 2: The estimate of $|(\nabla_\xi X^s, \nabla_\xi \Xi^s)|$ for $0 \leqslant s \leqslant s_1$.

Firstly, taking ∇_ξ to (1.7) and integrating the resulting equation over $[0, s]$, we arrive at

$$\begin{cases} \nabla_\xi X^s = \int_0^s \nabla_\xi \Xi^t \, dt, \\ \nabla_\xi \Xi^s = \frac{1}{2} \int_0^s \nabla_X^2 n^2(X^t) \nabla_\xi X^t(x, \xi) \, ds + I. \end{cases} \tag{5.49}$$

Substituting the second formula of (5.49) to the first one, we obtain

$$\nabla_\xi X^s = \frac{1}{2} \int_0^s (s-t) \nabla_X^2 n^2(X^t) \nabla_\xi X^t \, dt + sI. \tag{5.50}$$

We claim that

$$\int_0^s (s-t) |\nabla_X^2 n^2(X^t)| \, dt \leqslant C, \quad 0 \leqslant s \leqslant s_1. \tag{5.51}$$

Integrating twice (5.43) on $[t, s_1]$, one has

$$|X^t|^2 \geqslant \frac{\tilde{\delta}_1}{2} (s_1 - t)^2 + |X^{s_1}|^2 + 2(s_1 - t) X^{s_1} \cdot \Xi^{s_1}.$$

Since $|(X^{s_1}, \Xi^{s_1})| < 2R_0$, we obtain for some $c_0 = c_0(R_0, \tilde{\delta}_1) > 0$ such that

$$1 + |X^t|^2 \geqslant c_0 (1 + |t - s_1|^2), \quad 0 \leqslant t \leqslant s_1, \tag{5.52}$$

uniformly in (x, ξ) and s_1. Therefore, by (1.4),

$$\int_0^s (s-t) |(\nabla_X^2 n^2)(X^t)| \, dt \leqslant C \int_0^s \frac{(s-t)}{(1 + |s_1 - t|^2)^{1 + \frac{\rho_0}{2}}} \, dt \leqslant C',$$

uniformly in $0 \leqslant s \leqslant s_1$. C' is independent of (x, ξ) and s_1. This proves (5.51), which together with (5.50) and Gronwall inequality implies that

$$|\nabla_\xi X^s| \leqslant C \langle s \rangle, \quad \forall 0 \leqslant s \leqslant s_1. \tag{5.53}$$

Substituting (5.53) to the second formula of (5.49), we obtain by the same argument that

$$|\nabla_\xi \Xi^s| \leqslant C \left(\int_0^s \frac{t}{(1 + |X^t|^2)^{1 + \frac{\rho_0}{2}}} \, dt + 1 \right) \leqslant C' \langle s \rangle, \tag{5.54}$$

for all $0 \leqslant s \leqslant s_1$.

Step 3: The estimate of $|(\nabla_\xi^\beta X^s, \nabla_\xi^\beta \Xi^s)|$ for $|\beta| > 1$.

We want to prove that

$$|(\nabla_\xi^\beta X^s, \nabla_\xi^\beta \Xi^s)| \leqslant C_\beta \langle s\rangle^{|\beta|} \tag{5.55}$$

for all $0 \leqslant s \leqslant s_1$, $|\beta| > 1$. Eq. (5.55) for $|\beta| = 1$ is proved in Step 2. Assume now that (5.55) is true for all $1 \leqslant |\beta| \leqslant k-1$. We are going to prove it for $|\beta| = k$, $k \geqslant 2$.

Again by (1.7), for $|\beta| \geqslant 2$, there holds

$$\begin{cases} \nabla_\xi^\beta X^s(x,\xi) = \displaystyle\int_0^s \nabla_\xi^\beta \Xi^t(x,\xi)\,dt, \\ \nabla_\xi^\beta \Xi^s(x,\xi) = \displaystyle\frac{1}{2}\int_0^s \nabla_\xi^\beta \Big(\nabla_X n^2(X^t(x,\xi))\Big)\,dt. \end{cases} \tag{5.56}$$

Then by (5.50), we obtain

$$\begin{aligned} \nabla_\xi^\beta X^s = &\sum_{\substack{2 \leqslant l \leqslant k, \sum \beta_i = \beta \\ |\beta_i| < |\beta|}} C_{\beta,\beta_1,\dots,\beta_l} \int_0^s (s-t)\nabla_X^{l+1} n^2(X^t)\nabla_\xi^{\beta_1} X^t \cdots \nabla_\xi^{\beta_l} X^t\,dt \\ &+ \int_0^s (s-t)\nabla_X^2 n^2(X^t)\nabla_\xi^\beta X^t\,dt. \end{aligned} \tag{5.57}$$

Here $\nabla_X^l n^2$ denotes the tensor of derivatives of order l of $n^2(X)$. Let $\rho_{l+1} = \frac{l+1+\rho_0}{2}$. By (1.4) with $\rho_0 > 0$, (5.52) and the inductive assumption, we obtain

$$\begin{aligned} &\left|\int_0^s (s-t)\nabla_X^{l+1} n^2(X^t)\nabla_\xi^{\beta_1} X^t \cdots \nabla_\xi^{\beta_l} X^t\,dt\right| \\ &\quad \leqslant C\int_0^s \frac{(s-t)\langle t\rangle^k}{(1+|X^t|^2)^{\rho_{l+1}}}\,dt \leqslant C_1\langle s\rangle^k \int_0^s \frac{(s-t)}{\langle s_1 - t\rangle^{\rho_{l+1}}}\,dt \leqslant C_2\langle s\rangle^k \end{aligned} \tag{5.58}$$

uniformly in $0 \leqslant s \leqslant s_1$.

Then by summing up (5.51), (5.57), (5.58), and applying Gronwall inequality on $[0, s_1]$, we obtain for $|\beta| = k$

$$|\nabla_\xi^\beta X^s| \leqslant C_k \langle s\rangle^k \tag{5.59}$$

for all $0 \leqslant s \leqslant s_1$. And a similar proof of (5.59) also yields the same estimate for $|\nabla_\xi^\beta \Xi^s|$ which proves (5.55) for $|\beta| = k$. By induction, (5.55) is proved for all β. The same method can be used to obtain global estimates on the x-derivatives of (X^s, Ξ^s), but for the proof of Lemma 5.3, we only need their derivatives in ξ.

Step 4: The proof of (5.39) for the general case.

For $R \in C_0^\infty(\Omega)$, and for any (x, ξ) and s such that $(X^s, \Xi^s) \in \operatorname{supp} R$, one has $|s - s_1| \leqslant T_1$ where T_1 depends only on $\operatorname{supp} R$. Then, (5.55) implies that

$$|\partial_\xi^\beta R^s(x, \xi)| \leqslant C_\beta \langle s\rangle^{|\beta|} \tag{5.60}$$

uniformly (x, ξ) and $s \geqslant 0$. With (5.46) and (5.60), for any multi-index β with $|\beta| \leqslant M$, we obtain,

$$\begin{aligned}
&|y^\beta \widehat{g}(x, y)| \\
&\quad = \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \int_{|\xi| \leqslant R_0} \int_0^\infty e^{-iy\xi} \exp(-\alpha s) D_\xi^\beta R^s(x, \xi)\, ds\, d\xi \right| \\
&\quad \leqslant C \int_{|\xi| \leqslant R_0} \int_{s_0}^{s_0+T_1} \exp(-\alpha s) \langle s\rangle^M \, ds\, d\xi \\
&\quad \leqslant C \langle x\rangle^M \exp(-\alpha|x|),
\end{aligned} \tag{5.61}$$

which together with (5.48) proves (5.39) for integer M case. With a simple interpolation argument, we can prove (5.39) for general positive number M.

Eq. (5.40) can be proved in the same way. The details are omitted. □

Acknowledgments

This research is motivated by a lecture of F. Castella on the paper [5] given in Nantes, October 2002. The authors thank F. Castella for interesting discussions on the topic. They also thank the referee for many concrete and helpful suggestions which improved this work, and Zhifei Zhang for having pointed out an error related to Theorem 4.1 in an early version. X.P. Wang is partially supported by a grant of Outstanding Overseas Chinese Scholars of the Chinese Academy of Sciences, and P. Zhang is partially supported by the NSF of China under Grant 10131050 and 10276036, and the innovation grant from the Chinese Academy of Sciences.

References

[1] S. Agmon, L. Hörmander, Asymptotic properties of solutions to differential equations with simple characteristics, J. Anal. Math. 30 (1976) 1–38.

[2] J.D. Benamou, F. Castella, T. Katsaounis, B. Perthame, High frequency limit of the Helmholtz equation, Rev. Mat. Iberoamericana 18 (2002) 187–209.

[3] N. Burq, Mesures semi-classiques et mesures de défaut, Séminaire Bourbaki, No. 826, 1996–1997.

[4] F. Castella, The radiation condition at infinity for the high frequency Helmholtz equation with source term: a wave packet approach, J. Funct. Anal. 223 (2005) 204–257.

[5] F. Castella, B. Perthame, O. Runborg, High frequency limit of the Helmholtz equation II: source on a general smooth manifold, Comm. Partial Differential Equations 27 (2002) 607–651.

[6] C. Gérard, A. Martinez, Principe d'absorption limite pour des opéateurs de Schrödinger *à* longue portée, C. R. Acad. Sci. Paris Sér. I Math. 306 (3) (1988) 121–123.
[7] P. Gérard, Microlocal defect measures, Comm. Partial Differential Equations 16 (1991) 1761–1794.
[8] P. Gérard, Mesures semi-classiques et ondes de Bloch, Séminaire sur les équations aux Dérivées Partielles, 1990–1991, École Polytech., Palaiseau, 1991.
[9] L. Hörmander, The Analysis of Linear Partial Differential Operator II, Springer, Berlin, 1983.
[10] A. Jensen, P. Perry, Commutator methods and Besov space estimates for Schrödinger operators, J. Operator Theory 14 (1985) 181–188.
[11] P.L. Lions, T. Paul, Sur les measure de Wigner, Rev. Mat. Iberoamericana 9 (1993) 553–618.
[12] V.P. Maslov, M.V. Fedoriuk, Semi-classical Approximation in Quantum Mechanics, D. Reidel, Dordrecht, 1981.
[13] E. Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. 78 (1981) 391–408.
[14] E. Mourre, Opérateurs conjugués et propriétés de propagation, Comm. Math. Phys. 91 (1983) 279–300.
[15] B. Perthame, L. Vega, Morrey–Campanato estimates for Helmholtz equations, J. Funct. Anal. 164 (2) (1999) 340–355.
[16] D. Robert, H. Tamura, Semi-classical estimates for resolvents and asymptotics for total scattering cross sections, Ann. Inst. H. Poincaré 46 (1987) 415–442.
[17] L. Tartar, H-measure, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proc. Roy. Soc. Edinburgh Section A 115 (1990) 193–230.
[18] X.P. Wang, Time-decay of scattering solutions and resolvent estimates for semi-classical Schrödinger operators, J. Differential Equations 71 (1988) 348–395.
[19] X.P. Wang, Semiclassical resolvent estimates for N-body Schrödinger operators, J. Funct. Anal. 97 (1991) 466–483.
[20] X.P. Wang, Microlocal resolvent estimates for N-body Schrödinger operators, J. Faculty Sci., Univ. Tokyo Section IA 40 (1993) 337–385.
[21] E. Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev. 40 (1932) 742–759.
[22] B. Zhang, Commutator estimates, Besov spaces and scattering problems for the acoustic wave propagation in perturbed stratified fluids, Math. Proc. Cambridge Philos. Soc. 128 (1) (2000) 177–192.
[23] P. Zhang, Wigner measure and the semi-classical limit of Schrödinger–Poisson equations, SIAM J. Math. Anal. 34 (2002) 700–718.
[24] P. Zhang, Y. Zheng, N.J. Mauser, The limit from the Schrödinger–Poisson to the Vlasov–Poisson equations with general data in one dimension, Comm. Pure Appl. Math. 55 (2002) 582–632.